Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 13 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	15	
4	20	
5	25	
Total	100	

1. (20 total points) Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

(a) (5 points) Given that $x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $u(k) = 0$ for all k, determine $x(10)$.

(b) (10 points) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}$ and $x(0) = 0$.

(c) (5 points) Determine the stability of the system if $a = 0.8$ (asymptotically stable, marginally stable, or unstable?). What happens to $x(n)$ as $n \to \infty$?

2. (20 total points) Consider the following CT-LTV system

$$\dot{x}(t) = \begin{bmatrix} -\dfrac{1}{1+e^t} & \dfrac{1}{1+e^t} \\ 0 & \dfrac{-2}{1+e^t} \end{bmatrix} x(t).$$

   (a) (20 points) Find the state transition matrix of the above system.

Hint: $\displaystyle\int \frac{1}{1+e^t}dt = \int \frac{1+e^t-e^t}{1+e^t}dt = \int \frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t}dt = ??$

3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

   (a) (5 points)

$$x(k+1) = \begin{bmatrix} 0.999 & 0 \\ 0 & -2 \end{bmatrix} x(k)$$

   (b) (5 points)

$$\dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} u(t)$$

   (c) (5 points)

$$\dot{x}(t) = -0.0000001x(t) + 100000u(t)$$

4. (20 total points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1)\\ \dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3\end{aligned}$$

   (a) (10 points) The above dynamic system has **many** equilibrium points. Find exactly **two** distinct equilibrium points.

(b) (10 points) Analyze the local stability of the system by obtaining the linearized models around these two equilibrium points.

5. (25 total points) The following CT LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} -3t^2 & \sin(t) & 0 \\ 0 & -3t^2 & 0 \\ 0 & 0 & -15\cos(0) \end{bmatrix} x(t).$$

(a) (15 points) Obtain the state transition matrix $\phi(t, t_0)$ for the above system. To receive full credit, you have to clearly show your steps. You do not have to multiply the individual matrices at the end of the STM computations.

(b) (5 points) Find $x(t)$ if $x(t_0 = 2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(c) (5 points) For any initial condition $x(t_0)$, is the time-varying system asymptotically stable or unstable?

6. (10 total points) **Bonus Question:** Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system.

   This is a true/false question, so for you to get the full credit, you have to have the right answer for **all** T/F questions. For example, if you answer 4/5 correctly, you'll get 0 points. You don't have to prove your answer. Fill your answer in the box below.

   (a) (2 points) The system $\dot{x}(t) = -2Ax(t)$ is asymptotically stable.

   (b) (2 points) The system $\dot{x}(t) = (A^\top)^{-1}x(t)$ is asymptotically stable.

   (c) (2 points) The system $\dot{x}(t) = -A^{-1}x(t)$ is asymptotically stable.

   (d) (2 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

   (e) (2 points) The system $\dot{x}(t) = A^2x(t)$ is asymptotically stable.

   Your answer (a five-character string of T/F):