Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 6 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	15	
4	20	
5	25	
Total	100	

1. (20 total points) Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

(a) (5 points) Given that $x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $u(k) = 0$ for all k, determine $x(10)$.

$$x(k+1) = Ax(k) \Rightarrow x(10) = A^1 0x(0) \Rightarrow x(10) = \begin{bmatrix} 10a^{10-1} & 1 \\ 0 & a^{10} \end{bmatrix} x(0)$$

$$\Rightarrow x(10) = \begin{bmatrix} 10a^{10-1} & 1 \\ 0 & a^{10} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 20a^9 - 2 \\ -2a^{10} \end{bmatrix}$$

(b) (10 points) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}$ and $x(0) = 0$.

$$x(n) = \sum_{k=0}^{n-1} A^{n-1-k} Bu(k) = \sum_{k=0}^{n-1} A^k Bu(n-1-k) = \sum_{k=0}^{n-1} A^k B a^{k-n+1} = \sum_{k=0}^{n-1} \begin{bmatrix} ka^{k-1}a^{k-n+1} \\ 0 \end{bmatrix}.$$

Hence,

$$x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} a^{-n+2} \sum_{k=0}^{n-1} k(a^2)^k \\ 0 \end{bmatrix} = \begin{bmatrix} a^{-n+2} \frac{d}{da^2} \left(\frac{1-(a^2)^n}{1-a^2} \right) \\ 0 \end{bmatrix}.$$

(c) (5 points) Determine the stability of the system if $a = 0.8$ (asymptotically stable, marginally stable, or unstable?). What happens to $x(n)$ as $n \to \infty$?

If $a = 0.8$, we obtain:

$$\lim_{n\to\infty} x(n) = \lim_{a=0.8, n\to\infty} \begin{bmatrix} a^{-n+2} \frac{d}{da^2} \left(\frac{1-(a^2)^n}{1-a^2} \right) \\ 0 \end{bmatrix} =$$

$$\lim_{a=0.8, n\to\infty} \begin{bmatrix} a^{-n+2} \frac{1-n(a^2)^{n-1}+(n-1)(a^2)^n}{(1-a^2)^2} \\ 0 \end{bmatrix} = \begin{bmatrix} \infty \\ 0 \end{bmatrix}.$$

Hence, the system is unstable, since $\lim_{n\to\infty} n\alpha^n = 0$ if $|\alpha| < 1$, and in this case, $\alpha = a^2 = 0.8^2 < 1$ ($\lim_{n\to\infty} a^{2-n} = \infty$).

2. (20 total points) Consider the following CT-LTV system

$$\dot{x}(t) = \begin{bmatrix} -\dfrac{1}{1+e^t} & \dfrac{1}{1+e^t} \\ 0 & \dfrac{-2}{1+e^t} \end{bmatrix} x(t).$$

   (a) (20 points) Find the state transition matrix of the above system.

Hint: $\displaystyle\int \frac{1}{1+e^t}dt = \int \frac{1+e^t-e^t}{1+e^t}dt = \int \frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t}dt = ??$

- First, note that

$$\int_{t_0}^{t} \frac{1}{1+e^q}dq = \int_{t_0}^{t} \frac{1+e^q}{1+e^q} - \frac{e^q}{1+e^q}dq = t - t_0 - \ln\left(\frac{1+e^{t_0}}{1+e^t}\right) = \beta(t,t_0)$$

- Furthermore, $A(t)$ can be written as

$$A(t) = \frac{1}{1+e^t}\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \frac{1}{1+e^t}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}.$$

- Hence, we can write $A(t)$ as

$$A(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\dfrac{1}{1+e^t} & 0 \\ 0 & -2\dfrac{1}{1+e^t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}$$

  implying that

$$\phi(t,t_0) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\beta(t,t_0) & 0 \\ 0 & -2\beta(t,t_0) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}$$

  where

$$\beta(t,t_0) = t - t_0 - \ln\left(\frac{1+e^{t_0}}{1+e^t}\right).$$

3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

   (a) (5 points)

$$x(k+1) = \begin{bmatrix} 0.999 & 0 \\ 0 & -2 \end{bmatrix} x(k)$$

Unstable, since $eig(A) = \{0.999, -2\}$ and $|\lambda_2| = 2 > 1$ is outside the unit disk.

   (b) (5 points)

$$\dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 0 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} u(t)$$

Marginally stable, since $eig(A) = \{-0.4, -0.4, -0.4, 0\}$. Hence, all eigenvalues of A are in the closed LHP, and the evalue on the $j\omega$ axis has Jordan block of size 1. Hence, the system is marginally stable.

   (c) (5 points)

$$\dot{x}(t) = -0.0000001 x(t) + 100000 u(t)$$

System is asymptotically stable, as the only evalue for this system is $\lambda = -0.0000001$ which is still in the open LHP.

4. (20 total points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t)-1)\\ \dot{x}_2(t) &= x_2^2(t)+x_1(t)-3\end{aligned}$$

(a) (10 points) The above dynamic system has **many** equilibrium points. Find exactly **two** distinct equilibrium points.

Two equilibrium points for the above system are

- $x^{eq_1} = \begin{bmatrix}1\\ \sqrt{2}\end{bmatrix}$
- $x^{eq_2} = \begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix}$. Other equilibrium points:
- $x^{eq_3} = \begin{bmatrix}-1\\ 2\end{bmatrix}$
- $x^{eq_4} = \begin{bmatrix}3\\ 0\end{bmatrix}$

(b) (10 points) Analyze the local stability of the system by obtaining the linearized models around these two equilibrium points.

Local dynamics of the system can be written as

$$\Delta\dot{x}(t) = A^{eq_1}\Delta x(t)$$

or

$$\Delta\dot{x}(t) = A^{eq_2}\Delta x(t)$$

where $\Delta x(t) = x(t) - x^{eq_1}$ or $\Delta x(t) = x(t) - x^{eq_2}$. Matrix A is equal to

$$A = \begin{bmatrix}2x_1x_2 & x_1^2-1\\ 1 & 2x_2\end{bmatrix}$$ **evaluated at the equilibrium points**.

Hence,

- For $x^{eq_1} = \begin{bmatrix}1\\ \sqrt{2}\end{bmatrix}$, $A = \begin{bmatrix}2\sqrt{2} & 0\\ 1 & 2\sqrt{2}\end{bmatrix}$. This matrix is not stable seeing that the two evalues (on the diagonal) are both positive. Hence, the system is unstable locally around x^{eq_1}.
- $x^{eq_2} = \begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix}$, $A = \begin{bmatrix}-2\sqrt{2} & 0\\ 1 & -2\sqrt{2}\end{bmatrix}$. This matrix is stable seeing that the two evalues (on the diagonal) are both negative. Hence, the system is asymptotically stable locally around x^{eq_2}.

5. (25 total points) The following CT LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} -3t^2 & \sin(t) & 0 \\ 0 & -3t^2 & 0 \\ 0 & 0 & -15\cos(0) \end{bmatrix} x(t).$$

(a) (15 points) Obtain the state transition matrix $\phi(t, t_0)$ for the above system. To receive full credit, you have to clearly show your steps. You do not have to multiply the individual matrices at the end of the STM computations.

- Notice that

$$A(t) = \begin{bmatrix} A_{11}(t) & \\ & A_{22}(t) \end{bmatrix}, A_{11}(t) = \begin{bmatrix} -3t^2 & \sin(t) \\ 0 & -3t^2 \end{bmatrix}, A_{22}(t) = \begin{bmatrix} -15\cos(0) = -15 \end{bmatrix}.$$

- Hence, the problem simplifies to finding two STMs for $A_{11}(t)$ and $A_{22}(t)$ since $A(t)$ is block diagonal.
- STM of $A_{22}(t)$:

$$\phi_{22}(t, t_0) = e^{-15(t-t_0)}.$$

- STM of $A_{11}(t)$:
  Notice that

$$A_{11}(t) = -3t^2 I_2 + \sin(t) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent of order 2. Hence, we can write:

$$\phi_{22}(t, t_0) = e^{\int_{t_0}^{t} -3\tau^2 d\tau} \left(I_2 + (\cos(t_0) - \cos(t)) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = e^{-t^3 + t_0^3} \begin{bmatrix} 1 & \cos(t_0) - \cos(t) \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \phi_A(t, t_0) = \begin{bmatrix} e^{-t^3 + t_0^3} \begin{bmatrix} 1 & \cos(t_0) - \cos(t) \\ 0 & 1 \end{bmatrix} & \\ & e^{-15(t-t_0)} \end{bmatrix}.$$

(b) (5 points) Find $x(t)$ if $x(t_0 = 2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$x(t) = \phi_A(t,t_0)x(t_0) = \begin{bmatrix} e^{-t^3+t_0^3}\begin{bmatrix}1 & \cos(t_0)-\cos(t) \\ 0 & 1\end{bmatrix} & \\ & e^{-15(t-t_0)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-t^3+t_0^3}(1+\cos(t)-\cos(t_0)) \\ -e^{-t^3+t_0^3} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t^3+8}(1+\cos(t)-\cos(2)) \\ -e^{-t^3+8} \\ 0 \end{bmatrix}$$

(c) (5 points) For any initial condition $x(t_0)$, is the time-varying system asymptotically stable or unstable?

Any TV-LTI system is asymptotically stable if and only if $\lim_{t\to\infty} \phi_A(t,t_0) = 0-$matrix. By evaulating the limit in this problem we get this result, hence the system is asymptotically stable seeing that $\phi_A(\infty, t_0) = 0$.

6. (10 total points) **Bonus Question:** Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system.

This is a true/false question, so for you to get the full credit, you have to have the right answer for **all** T/F questions. For example, if you answer 4/5 correctly, you'll get 0 points. You don't have to prove your answer. Fill your answer in the box below.

(a) (2 points) The system $\dot{x}(t) = -2Ax(t)$ is asymptotically stable.

(b) (2 points) The system $\dot{x}(t) = (A^\top)^{-1}x(t)$ is asymptotically stable.

(c) (2 points) The system $\dot{x}(t) = -A^{-1}x(t)$ is asymptotically stable.

(d) (2 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

(e) (2 points) The system $\dot{x}(t) = A^2x(t)$ is asymptotically stable.

Your answer (a five-character string of T/F):

F-T-F-F-F