Your Name:

Your Signature:

- **Exam duration:** 1 hour and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	35	
2	20	
3	25	
4	20	
Total	100	

1. (35 total points) You are given the following LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) What are the eigenvalues of the system? And Is the system stable?

The eigenvalues of A are: $eig(A) = \{1, 0, -1\}$. The eigenvalues 0, 1 are not asymptotically stable, hence the system is unstable.

(b) (5 points) Is the above system controllable or not? Justify your answer.

The controllability matrix can be computed as follows:

$$\mathcal{C} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is a rank 2 matrix. Hence, the system is not controllable.

(c) (5 points) What is the controllability subspace?

The controllability subspace is the range-space of the controllability matrix, which can be written as:

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

(d) (5 points) Is the above system observable or not? Justify your answer.

The observability matrix can be computed as:

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and is a rank-1 matrix. Hence, the system is not observable.

(e) (5 points) Obtain the unobservable subspace of the system.

The unobservable subspace of the system is the null space of the observability matrix:

$$Null(\mathcal{O}) = Null\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\right),$$

which can be written as the span of these two vectors:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(f) (5 points) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -2, -2\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

No. There isn't. As eigenvalue $\lambda = -1$ fails the PBH test, this means this evalue will be present in the closed loop system dynamics no matter what kind of linear state feedback we employ.

(g) (5 points) Is there a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

No. There isn't. The two evalues 0, 1 both fail the PBH test, which means the system is not detectable. This means that no matter what kind of linear observer we design, these two evalues will be present in the closed loop dynamics. Hence, it's not possible to design a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$.

2. (20 total points) The following LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{1+e^t} & \frac{1}{1+e^t} & 0 \\ 0 & \frac{-2}{1+e^t} & 0 \\ 0 & 0 & 10 \end{bmatrix} x(t).$$

Hint:

$$\int_{t_0}^{t} \frac{1}{1+e^t} dt = \int_{t_0}^{t} \frac{1+e^t-e^t}{1+e^t} dt = \int_{t_0}^{t} \left(\frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t} \right) dt$$

(a) (20 points) Find the state transition matrix of this system.

- First, note that the STM of the system is:

$$\phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \\ & \phi_{22}(t,t_0) \end{bmatrix},$$

where $\phi_{22}(t,t_0)$ is the state transition matrix of $A_{22} = 10$ and $\phi_{11}(t,t_0)$ is the STM of $A_{11}(t) = \begin{bmatrix} -\frac{1}{1+e^t} & \frac{1}{1+e^t} \\ 0 & \frac{-2}{1+e^t} \end{bmatrix}$.

- Given that, it's easy to see that $\phi_{22}(t,t_0) = e^{10(t-t_0)}$.
- To find $\phi_{11}(t,t_0)$, we can first write $A_{11}(t)$ as:

$$A_{11}(t) = \begin{bmatrix} -\frac{1}{1+e^t} & \frac{1}{1+e^t} \\ 0 & \frac{-2}{1+e^t} \end{bmatrix} = \frac{1}{1+e^t} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \frac{1}{1+e^t} \tilde{A}_{11}.$$

- Notice that:

$$\int_{t_0}^{t} \frac{1}{1+e^t} dt = \int_{t_0}^{t} \frac{1+e^t-e^t}{1+e^t} dt = \int_{t_0}^{t} \left(\frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t} \right) dt$$

$$= t - t_0 + \ln(1+e^{t_0}) - \ln(1+e^t) = t - t_0 + \ln\left(\frac{1+e^{t_0}}{1+e^t}\right) = f(t,t_0).$$

- To find $\phi_{11}(t,t_0)$, we should diagonalize $\tilde{A}_{11}$:

$$\tilde{A}_{11} = TDT^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1},$$

hence

$$e^{\tilde{A}_{11} f(t,t_0)} = TD(t,t_0)T^{-1} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} & \\ & e^{-2\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1}.$$

- Finally, we can write the STM of $A(t)$ as:

$$\phi(t,t_0) = \begin{bmatrix} T \begin{bmatrix} e^{-\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} & \\ & e^{-2\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} \end{bmatrix} T^{-1} & \\ & e^{10(t-t_0)} \end{bmatrix},$$

where $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

- Note that

$$e^{-\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} = \frac{e^{t_0}}{1+e^{t_0}}\left(e^{-t}+1\right)$$

and

$$e^{-2\left(t-t_0+\ln\left(\frac{1+e^{t_0}}{1+e^t}\right)\right)} = \left(\frac{1+e^t}{1+e^{t_0}}\right)^2 e^{-2t+2t_0} = \left(e^{-2t}+2e^{-t}+1\right)\frac{e^{2t_0}}{(1+e^{t_0})^2}.$$

Hence, the STM of $A(t)$ can be written compactly as follows:

$$\phi(t,t_0) = \begin{bmatrix} T \begin{bmatrix} \frac{e^{t_0}}{1+e^{t_0}}\left(e^{-t}+1\right) & \\ & \left(e^{-2t}+2e^{-t}+1\right)\frac{e^{2t_0}}{(1+e^{t_0})^2} \end{bmatrix} T^{-1} & \\ & e^{10(t-t_0)} \end{bmatrix},$$

where $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

3. (25 total points) You are given the following SISO system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} x(t).$$

(a) (20 points) Design an observer based controller (i.e., $u(t) = -K\hat{x}(t)$) for the above system such that the desired eigenvalues for the closed loop system are all at $\lambda_{cl} = -5$ (i.e., for both $A - LC$ and $A - BK$).

First, you'll have to check if the system is controllable and observable (or detectable and stabilizable).

- First, note that the system is both controllable and observable as the rank of the controllability and observability matrices are both equal to 2.
- Second, we design a controller such that $A - BK$ has evalues at $\lambda = -5$. Similar to the examples from the homework, the state feedback controller is $K = [3.75 \;\; 3]$.
- Third, we design the observer. We find $A - LC$ in terms of l_1 and l_2:

$$A - LC = \begin{bmatrix} 1 - l_1/2 & 3 - l_1 \\ 3 - l_2/2 & 1 - l_2 \end{bmatrix}.$$

Since the roots of the designed observer are $-5, -5$, the desired characteristic polynomial is:

$$\pi_{A-LC} = (\lambda + 5)(\lambda + 5) = \lambda^2 + 10\lambda + 25.$$

The characteristic polynomial in terms of l_1 and l_2 can be written as:

$$+\lambda^2 + \lambda \underbrace{\left(-2 + \frac{l_1}{2} + l_2\right)}_{=10} \underbrace{-8 + \frac{5l_1}{2} + \frac{l_2}{2}}_{=25} = 0.$$

Solving the following linear system of equations,

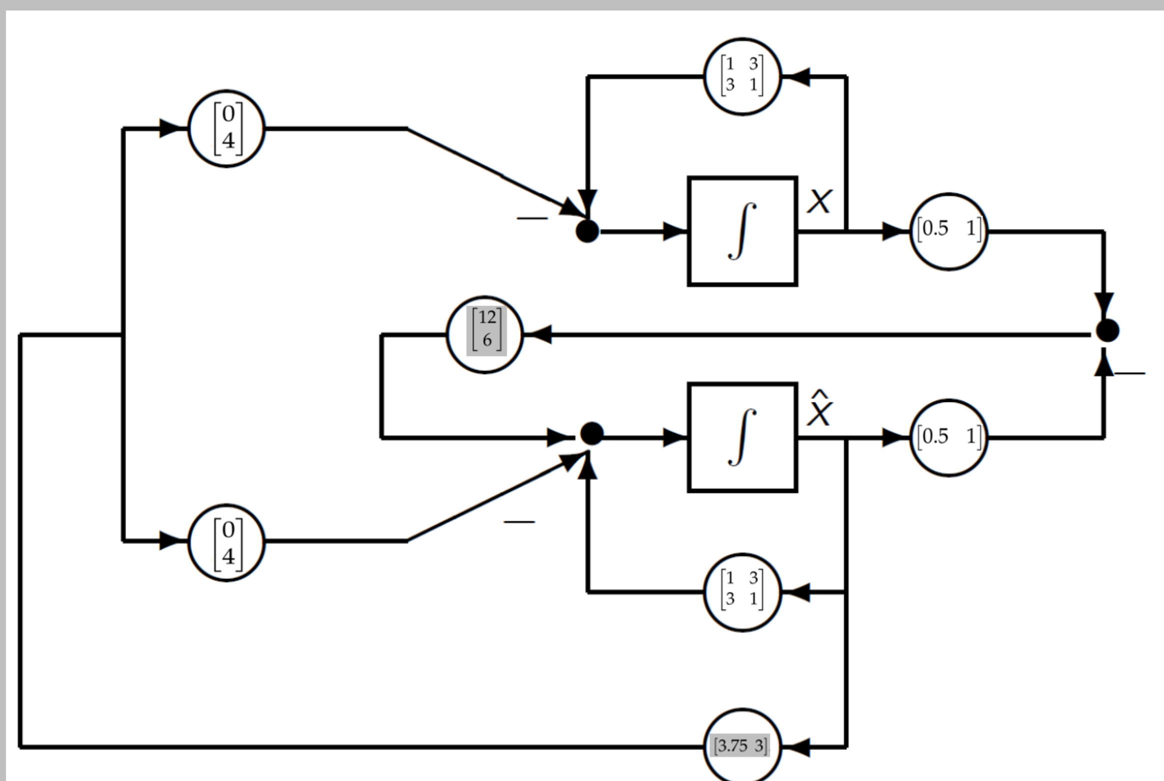
$$\begin{aligned} 25 &= -8 + \frac{5l_1}{2} + \frac{l_2}{2} \\ 10 &= -2 + \frac{l_1}{2} + l_2, \end{aligned}$$

we obtain $l_1 = 12$ and $l_2 = 6$, $\Rightarrow l = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

- Finally, overall system design:

$$\begin{aligned} u(t) &= -K\hat{x}(t) = -3.75\hat{x}_1(t) - 3\hat{x}_2(t) \\ \dot{\hat{x}}_1(t) &= \hat{x}_1(t) + 3\hat{x}_2(t) + 12(y(t) - \hat{y}(t)) \\ \dot{\hat{x}}_2(t) &= 3\hat{x}_1(t) + \hat{x}_2(t) + 4u(t) + 6(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= 0.5\hat{x}_1(t) + \hat{x}_2(t) \end{aligned}$$

(b) (5 points) Draw a block diagram representation of the overall system with the observer based controller, including the values for the gains K and L that you have designed.



4. (20 total points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t)\\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1.\end{aligned}$$

   (a) (10 points) Find all the equilibrium points of the nonlinear system. You should find at least 2 equilibrium points.

   The equilibrium points for this system are:

   - $x_e^{(1)} = \begin{bmatrix}3\\2\end{bmatrix}, x_e^{(2)} = \begin{bmatrix}0\\-1\end{bmatrix}.$

   (b) (10 points) Determine the stability of the system around each equilibrium point.

   The linearized system can be written as:

$$\Delta x(t) = \begin{bmatrix}x_{2e}-2 & x_{1e}\\ 1 & -1\end{bmatrix}\Delta x(t).$$

   For the first equilibrium point:

$$\Delta x(t) = \begin{bmatrix}0 & 3\\ 1 & -1\end{bmatrix}\Delta x(t),$$

   we obtain $eig(A) = \{1.3, -2.3\}$, hence the system is unstable around $x_e^{(1)}$. For the second equilibrium point:

$$\Delta x(t) = \begin{bmatrix}-3 & 0\\ 1 & -1\end{bmatrix}\Delta x(t),$$

   we obtain $eig(A) = \{-3, -1\}$, hence the system is asymptotically stable around $x_e^{(2)}$.