Your Name: ____________________ Your Signature: ____________________

- **Exam duration:** 3 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If I find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place [a box around your final answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 11 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	30	
2	25	
3	35	
4	30	
5	20	
6	15	
7	25	
8	20	
Total	200	

1. (30 total points) You are given the following LTI dynamical system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

where

$$A = \begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -1\end{bmatrix}, B = \begin{bmatrix}1\\0\\0\end{bmatrix}, C = \begin{bmatrix}1 & 0 & 0\end{bmatrix}.$$

(a) (5 points) What are the modes/eigenvalues of A? Is the system stable?

- The system modes are $\text{eig}(A) = \{1, -2, -1\}$.
- The system is unstable.

(b) (5 points) Is the above system controllable or not? Justify your answer.

- The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix}B & AB & A^2B\end{bmatrix} \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

- The rank of the controllability matrix is one, hence the system is not controllable.

(c) (5 points) Is the above system observable or not? Justify your answer.

- The observability matrix is:

$$\mathcal{O} = \begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$$

- The observability matrix is full rank, hence the system is observable.

(d) (5 points) Obtain the unobservable subspace of the system—if it exists.

- Since the system is fully observable, then the unobservable subspace is empty as the null-space of $\mathcal{O}$ is empty.

(e) (5 points) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -1, -3\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

- Yes, since the two uncontrollable eigenvalues—that lead to rank-1 $\mathcal{C}$—are the stable ones (evalues $-1, -2$ both fail the PBH test).
- $K = \begin{bmatrix} 4 & 1 & 3.82 \end{bmatrix}$.

(f) (5 points) Is there a state observer such that $A - LC$ has eigenvalues $\{-4, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

- Yes, since the system is observable.
- $L = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$.

2. (25 total points) The following LTV system is given:

$$\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -\alpha + \beta\cos(t) & -3 \\ 3 & -\alpha + \beta\cos(t) \end{bmatrix} x(t).$$

(a) (10 points) First, find the matrix exponential of this matrix for any real a and b:

$$A_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

- A_1 can be written as:

$$A_1 = aI_2 + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}.$$

- Hence, we can write

$$e^{A_1(t-t_0)} = e^{a(t-t_0)} \begin{bmatrix} \cos(b(t-t_0)) & \sin(b(t-t_0)) \\ -\sin(b(t-t_0)) & \cos(b(t-t_0)) \end{bmatrix}.$$

- We derived this result in Homework 3, Problem 12.

(b) (15 points) Use the answer in the previous part to find the state-transition matrix of $A(t)$.

- We can write $A(t)$ as

$$A(t) = \begin{bmatrix} -\alpha + \beta\cos(t) & -3 \\ 3 & -\alpha + \beta\cos(t) \end{bmatrix} = \begin{bmatrix} -\alpha & -3 \\ 3 & -\alpha \end{bmatrix} + \beta\cos(t)I_2.$$

- Hence, the state-transition matrix of $A(t)$ can be written as:

$$\phi_A(t,t_0) = e^{\beta(sin(t)-sin(t_0))}e^{-\alpha(t-t_0)} \begin{bmatrix} \cos(-3(t-t_0)) & \sin(-3(t-t_0)) \\ -\sin(-3(t-t_0)) & \cos(-3(t-t_0)). \end{bmatrix}$$

3. (35 total points) You are given the following SISO system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

(a) (20 points) Design an observer-based controller (i.e., $u(t) = -K\hat{x}(t)$) for the above system such that the desired eigenvalues for the closed loop system are all at $\lambda_{cl} = \{-2, -3\}$ for both the controller and the observer.

First, you'll have to check if the system is controllable and observable (or detectable and stabilizable).

- The system is controllable as the controllability matrix $\mathcal{C} = \begin{bmatrix} 0 & 2 \\ 2 & 8 \end{bmatrix}$ is full rank.
- In addition, the system is observable as the observability matrix $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ is full rank. Hence, we can design an observer-based controller with arbitrary closed-loop system eigenvalues.
- First, we design the state feedback controller $u(t) = -K\hat{x}(t)$ such that $\text{eig}(A - BK) = \{-2, -3\}$. The K that does the job is $K = \begin{bmatrix} 0 & 3.5 \end{bmatrix}$ (the calculations are omitted here).
- Second, we design the Luenberger observer such that $\text{eig}(A - LC) = \{-2, -3\}$.The Luenberger gain that does the job is $L = \begin{bmatrix} 7 \\ 42 \end{bmatrix}$.

(b) (5 points) Draw a block diagram representation of the overall system with the observer based controller, including the values for the gains K and L that you have designed.

- See lecture notes.

(c) (10 points) Write a MATLAB code to simulate the observer-based controller you designed above.

- See homework solutions.

4. (30 total points) The nonlinear, spinning body dynamics of a satellite can be written as

$$\begin{aligned}\dot{\omega}_1(t) &= \frac{I_2 - I_3}{I_1}\omega_2(t)\omega_3(t) + \frac{1}{I_1}\tau_1(t)\\ \dot{\omega}_2(t) &= \frac{I_3 - I_1}{I_2}\omega_3(t)\omega_1(t) + \frac{1}{I_2}\tau_2(t)\\ \dot{\omega}_3(t) &= \frac{I_1 - I_2}{I_3}\omega_1(t)\omega_2(t) + \frac{1}{I_3}\tau_3(t)\end{aligned}$$

where $I_{1,2,3}$ are the moments of inertia about principal axes (and are constants); $\omega_{1,2,3}$ are the angular velocities about principal axes; $\tau_{1,2,3}$ are the torques and control inputs about principal axes.

(a) (5 points) Consider that the system states are the three angular velocities and that the control inputs are the three torques. What is a trivial equilibrium point (i.e., control inputs and state equilibrium points) of this system?

- The equilibrium point of the system is the trivial one:

$$\omega_{1,2,3}^{\text{equi.}} = \tau_{1,2,3}^{\text{equi.}} = 0$$

(b) (10 points) Obtain the linearized representation of the system around the trivial equilibrium point.

- Let

$$\Delta x(t) = \begin{bmatrix}\omega_1 - \omega_1^{\text{equi.}}\\ \omega_2 - \omega_2^{\text{equi.}}\\ \omega_3 - \omega_3^{\text{equi.}}\end{bmatrix}, \Delta u(t) = \begin{bmatrix}\tau_1 - \tau_1^{\text{equi.}}\\ \tau_2 - \tau_2^{\text{equi.}}\\ \tau_3 - \tau_3^{\text{equi.}}\end{bmatrix},$$

then we can write:

$$\Delta\dot{x}(t) = A\Delta x(t) + B\Delta u(t) = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\Delta x(t) + \begin{bmatrix}\frac{1}{I_1} & 0 & 0\\ 0 & \frac{1}{I_2} & 0\\ 0 & 0 & \frac{1}{I_3}\end{bmatrix}\Delta u(t)$$

(c) (5 points) Determine the stability of the system around the equilibrium point.

- The system is unstable around the trivial equilibrium point. The 0-evalue has a geometric multiplicity smaller than the algebraic multiplicity, which leads to Jordan block of size greater than one and an unstable system.

(d) (10 points) Is the linearized system controllable? Stabilizable? Justify your answer. You should give two solutions to this problem: the first based on the properties of controllability we discussed in class, and another solution based on the physical interpretation of the linearized dynamics.

- First solution: by computing the controllability matrix $\mathcal{C}$, one can see that that $\mathcal{C}$ is full-row rank, hence the system is controllable and therefore stabilizable.
- Second solution: by simply observing the linearized system dynamics, we see that each control input can impact each state in a decoupled way. Hence, the states of the linearized system can be steered arbitrarily by the means of specific control actions. Hence, the linearized system is controllable and stabilizable.

5. (20 total points) *[You're halfway through the exam. You're getting there. Remember, look at the glass half-full, because emptiness is harder to quantify.]*

   Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

   (a) (20 points) Prove that the above system is controllable if the controllability matrix is full-rank.

   - Oh, you guys should know this.

6. (15 total points) Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_{t_0}.$$

(a) (15 points) Prove that the closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

Note that to prove that a certain function is a solution to any ODE, you have to prove that the initial conditions hold, and that the analytic solution is true for all $t > t_0$.

Hint — Leibniz Differentiation Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x + f\big(b(\theta),\theta\big)\cdot b'(\theta) - f\big(a(\theta),\theta\big)\cdot a'(\theta)$$

- First, note that the initial value satisfies the provided candidate solution:

$$x(t_0) = \underbrace{e^{A(t_0-t_0)}}_{I_n}x_{t_0} + \underbrace{\int_{t_0}^{t_0} e^{A(t-\tau)}Bu(\tau)\,d\tau}_{0} = x(t_0).$$

- Applying Leibniz Differentiation Theorem, we get (note that here, $b(\theta) \equiv t$ and $b'(\theta) = 1$):

$$\begin{aligned}
\dot{x}(t) &= Ae^{A(t-t_0)}x_{t_0} + \frac{d}{dt}\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \\
&= Ae^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} \frac{d}{dt}e^{A(t-\tau)}Bu(\tau)\,d\tau + e^{A(t-t)}Bu(t)\cdot 1 \\
&= Ae^{A(t-t_0)}x_{t_0} + A\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Bu(t) \\
&= A\left(e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau\right) + Bu(t) \\
&= Ax(t) + Bu(t).
\end{aligned}$$

- Therefore, the closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

7. (25 total points) Consider the following DT LTI system

$$x(k+1) = Ax(k) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} x(k), \; y(k) = Cx(k) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(k).$$

(a) (5 points) Is A nilpotent? Of what order?

- Yes; A is nilpotent of order 2 since $A^2 = 0$.

(b) (10 points) Suppose $y(0) = 1$ and $y(1) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

- We can write:

$$x(1) = Ax(0), \; y(0) = Cx(0), \; y(1) = Cx(1) = CAx(0)$$

- Hence,

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x(0) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(0).$$

- Since $\begin{bmatrix} C \\ CA \end{bmatrix}$ is nonsingular, we are able to uniquely determine $x(0)$ from the available measurements:

$$x(0) = \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

(c) (10 points) Suppose $y(1) = 1$ and $y(2) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

- We can write

$$y(2) = Cx(2) = CAx(1) = CA^2x(0) = 0$$

and

$$y(1) = Cx(1) = CAx(0) = 1.$$

- Note that since $CA^2 = 0$ (as A is nilpotent of order 2), and $y(2) = 0$, then any $x(0)$ is a solution to the equation $CA^2x(0) = 0$. In addition,

$$y(1) = CAx(0) = \begin{bmatrix} 1 & -2 \end{bmatrix} x(0) = 1$$

is a single equation with two unknowns. This equation has infinitely many solutions. Therefore, $x(0)$ cannot uniquely be determined from $y(1)$ and $y(2)$.

8. (20 total points) *[This is the final question of the exam. This painful experience is about to end. I told you to look at the glass half full.]*

   Consider the following system

$$
\begin{aligned}
\dot{x}(t) &= TJT^{-1}x(t) + Bu(t) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\
y(t) &= Cx(t) \\
&= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x(t).
\end{aligned}
$$

   (a) (10 points) Obtain necessary conditions on the entries of B such that only λ_1 is controllable. This means λ_2 is simply not controllable.

   I'm not giving you the eigenvectors because they're beautiful—they're given for a purpose.

- First, note that $T = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$ contain the right eigenvectors and the rows of T^{-1} contain the left eigenvectors $w_{1,2,3,4}$ (the first row of T^{-1} is $w_1^\top$.)
- The left eigenvectors are provided so that you can use the eigenvector test of controllability which states that λ_1 is controllable if $w_1^\top B \neq 0$.
- For controllability of λ_1, we require:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_1 \neq 0.$$

- For λ_2 to be uncontrollable, we require:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_3 = 0.$$

- Therefore, we require $b_1 \neq 0$ and $b_3 = 0$.
- Note that since λ_1 has a single Jordan block of size 2, this implies that only one eigenvector is the solution to the eigenvector of λ_1, and that's the reason why we took w_1 as the left eigenvector and not w_2, as w_2 is a generalized evector of λ_1.

(b) (10 points) Obtain necessary conditions on the entries of C such that only λ_2 is observable. This means λ_1 is simply not observable.

- The right eigenvectors are provided so that you can use the eigenvector test for observability which states that evalue λ_i is observable if $Cv_i \neq 0$.
- For observability of λ_2, we require:

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = c_2 + c_3 \neq 0.$$

- For λ_1 to be unobservable, we require:

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = c_1 + c_4 = 0.$$

- Hence, we require $c_2 \neq c_3$ and $c_1 = -c_4$.