Your Name: ____________________ Your Signature: ____________________

- **Exam duration:** 3 hours and 1 minute.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place [a box around your final answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 33 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	35	
2	25	
3	25	
4	15	
5	35	
6	25	
7	10	
8	20	
9	40	
9	40	
10	30	
Total	300	

1. (35 total points) You are given the following LTI dynamical system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

where

$$A = \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & -1\end{bmatrix}, B = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}, C = \begin{bmatrix}0 & 0 & 1\end{bmatrix}.$$

(a) (5 points) What are the eigenvalues of the system? And Is the system stable?

(b) (5 points) Is the above system controllable or not? Justify your answer.

(c) (5 points) Is the above system observable or not? Justify your answer.

(d) (10 points) Obtain the unobservable subspace of the system.

(e) (5 points) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -2, -2\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

(f) (5 points) Is there a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

2. (25 total points) The following LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} -t & 0 \\ -\cos(t) & -t \end{bmatrix} x(t) + \begin{bmatrix} \arccos(t^3) \\ \arctan(e^{\sin t}) \end{bmatrix} u(t).$$

   (a) (20 points) Find the state transition matrix of this LTV system.

(b) (5 points) Is this system asymptotically stable?

3. (25 total points) You are given the following SISO system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} x(t).$$

(a) (20 points) Design an observer based controller (i.e., $u(t) = -K\hat{x}(t)$) for the above system such that the desired eigenvalues for the closed loop system are all at $\lambda_{cl} = -5$ (i.e., for both $A - LC$ and $A - BK$).

First, you'll have to check if the system is controllable and observable (or detectable and stabilizable).

(b) (5 points) Draw a block diagram representation of the overall system with the observer based controller, including the values for the gains K and L that you have designed.

4. (15 total points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t)\\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1.\end{aligned}$$

   (a) (10 points) Find all the equilibrium points of the nonlinear system. You should find at least 2 equilibrium points.

(b) (5 points) Determine the stability of the system around each equilibrium point.

5. (35 total points) The MPC cost function minimization can be written as:

$$\underset{\Delta U}{\text{minimize}} \quad J(\Delta U) = \frac{1}{2}(r - Wx_a - Z\Delta U)^\top Q(r - Wx_a - Z\Delta U) + \frac{1}{2}\Delta U^\top R \Delta U,$$

where $r, W, x_a, Z, Q = Q^\top \succ 0, R = R^T \succ 0$ are all constant matrices and vectors with appropriate dimensions.

(a) (20 points) Obtain the optimal ΔU^*, the optimal solution for the above unconstrained problem.

(b) (5 points) Given a closed form solution for ΔU^*, show that ΔU^* is a minimizer to the above optimization problem. You have to prove that the second order necessary conditions are in fact satisfied.

(c) (10 points) Given part (a) of this problem, suppose that you're given the following constraints on the rate of change of the control action:

$$u^{\min} \leq \Delta U \leq u^{\max}.$$

Write the corresponding optimization problem in the following form:

$$\begin{array}{ll} \text{minimize} & J(\Delta U) \\ \text{subject to} & g(\Delta U) \leq 0\,, \end{array}$$

where $g(\Delta U)$ is a linear set of constraints that you need to write as linear inequality constraints.

6. (25 total points) You are given the following two matrices:

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \ P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

   (a) (25 points) Solve the above minimization problem for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\text{minimize} \quad x^\top Q x$$

$$\text{subject to} \quad x^\top P x = 1$$

   This problem might seem difficult, but it's not. You can simply expand this problem given Q and P are derive the KKT conditions.

7. (10 total points) Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system. For each of the following statements, determine if it is true or false. You do *not* have to justify your answer.

   (a) (2 points) The system $x(k+1) = -Ax(k)$ is asymptotically stable.

   (b) (2 points) The system $x(k+1) = A^\top x(k)$ is asymptotically stable.

   (c) (2 points) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume A^{-1} exists).

   (d) (2 points) The system $x(k+1) = (A + A^\top)x(k)$ is asymptotically stable.

   (e) (2 points) The system $x(k+1) = A^2x(k)$ is asymptotically stable.

8. (20 total points) Consider the following system

$$\begin{aligned}
\dot{x}(t) &= TJT^{-1}x(t) + Bu(t) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\
y(t) &= Cx(t) \\
&= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x(t).
\end{aligned}$$

(a) (10 points) Obtain necessary conditions on the entries of B such that only λ_1 is controllable. This means λ_2 is simply not controllable.

Hint: The eigenvector test does not change for systems having Jordan forms, i.e., you still apply the same test.

(b) (10 points) Obtain necessary conditions on the entries of C such that only λ_2 is observable. This means λ_1 is simply not observable.

9. (40 total points) Solve the following miscellaneous problems.

   (a) (15 points) Determine the values of α for which the function

   $$f(x,y,z) = 2x^2 + 2xz + 2\alpha yz + 2z^2$$

   is concave and the values for which it is convex.

(b) (15 points) Prove that the quadratic function $f(x) = x^\top Q x$ is convex if and only if $Q = Q^\top \succ 0$ (that is, Q is symmetric positive definite matrix).

(c) (10 points) Consider the following LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t).$$

where

$$A = \begin{bmatrix} 0.5a + 0.5c & 0 & 0.5a - 0.5c \\ 0 & b & 0 \\ 0.5a - 0.5c & 0 & 0.5a + 0.5c \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}.$$

This system has three eigenvalues/modes: $eig(A) = \{a, b, c\}$**.**

Which modes are **controllable**?

Which modes are **unobservable**?

10. (30 total points) This is a coding question.

    (a) (30 points) Write a MATLAB-based pseudo code that essentially solves a constrained MPC problem. The code should have the following **inputs**:

    - continuous time state-space matrices A, B, C defined by

    $$\dot{x} = Ax + Bu, \;\; y = Cx$$

      where matrices A, B, C are all arbitrary (i.e., of random dimensions)
    - initial conditions for the system (i.e., $x(0)$ and $u(-1) = 0$ if needed);
    - matrices Q and R defining weight matrices for the MPC problem;
    - reference signal to track r
    - input constraints on the control input given by

    $$u^{\min} \leq u(t) \leq u^{\max}$$

    - time horizon of T seconds, with a sampling time of 1 second;

    Your code should return the sequence of optimal control inputs $u(0), u(1), \ldots, u(T)$ as well as the states $x(1), \ldots, x(T+1)$. You should use the optimization method of MPC we learned in class where you construct $J(\Delta U)$ and then constrain it with the given constraint on $u(t)$. There are four important steps in your pseudo code:

    - Step 1: Obtaining discrete-time state space matrices.
    - Step 2: Finding the augmented MPC dynamics (i.e., $x_a(k)$ dynamics).
    - Step 3: Constructing matrices Z and W.
    - Step 4: Formulating the optimization problem

    $$\text{minimize} \;\; J(\Delta U) = \ldots \;\; ; \quad \text{subject to} \;\; g(\Delta U) = \ldots \leq 0$$

    - Step 5: Calling an optimization solver (you're free to use whatever you want).
    - Step 6: Extracting $u(0)$ from the solution and applying it to the system, that is, applying it to the ODE solver of $\dot{x} = Ax + Bu$.
    - Step 7: Doing the problem (Steps 1–6) in a while or a for loop until you get to the final time step. Note that some steps should not be in the for/while loop, seeing that some of them do not change (think computational efficiency). :-)