Your Name: | Your Signature:

- **Exam duration:** 3 hours and 1 minute.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 15 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	35	
2	25	
3	25	
4	15	
5	35	
6	25	
7	10	
8	20	
9	40	
9	40	
10	30	
Total	300	

1. (35 total points) You are given the following LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) What are the eigenvalues of the system? And Is the system stable?

The eigenvalues of A are: $eig(A) = \{1, 0, -1\}$. The eigenvalues $0, 1$ are not asymptotically stable, hence the system is unstable.

(b) (5 points) Is the above system controllable or not? Justify your answer.

The controllability matrix can be computed as follows:

$$\mathcal{C} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is a rank 2 matrix. Hence, the system is not controllable.

(c) (5 points) Is the above system observable or not? Justify your answer.

The observability matrix can be computed as:

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and is a rank-1 matrix. Hence, the system is not observable.

(d) (10 points) Obtain the unobservable subspace of the system.

The unobservable subspace of the system is the null space of the observability matrix:

$$Null(\mathcal{O}) = Null\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\right),$$

which can be written as the span of these two vectors:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(e) (5 points) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -2, -2\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

No. There isn't. As eigenvalue $\lambda = -1$ fails the PBH test, this means this evalue will be present in the closed loop system dynamics no matter what kind of linear state feedback we employ.

(f) (5 points) Is there a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

No. There isn't. The two evalues 0, 1 both fail the PBH test, which means the system is not detectable. This means that no matter what kind of linear observer we design, these two evalues will be present in the closed loop dynamics. Hence, it's not possible to design a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$.

2. (25 total points) The following LTV system is given:

$$\dot{x}(t) = \begin{bmatrix} -t & 0 \\ -\cos(t) & -t \end{bmatrix} x(t) + \begin{bmatrix} \arccos(t^3) \\ \arctan(e^{\sin t}) \end{bmatrix} u(t).$$

(a) (20 points) Find the state transition matrix of this LTV system.

Matrix $A(t)$ can be written as

$$A(t) = -tI_2 - \cos(t) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \beta_1(t)A_1 + \beta_2(t)A_2.$$

Note that $A_1A_2 = A_2A_1$ and $e^{A_2\gamma} = I_2 + A_2\gamma$ since A_2 is nilpotent of order 2. Hence,

$$\phi(t,t_0) = e^{t_0^2-t^2} \begin{bmatrix} 1 & 0 \\ \sin(t_0) - \sin(t) & 1 \end{bmatrix}$$

(b) (5 points) Is this system asymptotically stable?

Yes since the $\lim_{t\to\infty} \phi(t,t_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

3. (25 total points) You are given the following SISO system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} x(t).$$

(a) (20 points) Design an observer based controller (i.e., $u(t) = -K\hat{x}(t)$) for the above system such that the desired eigenvalues for the closed loop system are all at $\lambda_{cl} = -5$ (i.e., for both $A - LC$ and $A - BK$).

First, you'll have to check if the system is controllable and observable (or detectable and stabilizable).

- First, note that the system is both controllable and observable as the rank of the controllability and observability matrices are both equal to 2.
- Second, we design a controller such that $A - BK$ has evalues at $\lambda = -5$. Similar to the examples from the homework, the state feedback controller is $K = [3.75 \;\; 3]$.
- Third, we design the observer. We find $A - LC$ in terms of l_1 and l_2:

$$A - LC = \begin{bmatrix} 1 - l_1/2 & 3 - l_1 \\ 3 - l_2/2 & 1 - l_2 \end{bmatrix}.$$

Since the roots of the designed observer are $-5, -5$, the desired characteristic polynomial is:

$$\pi_{A-LC} = (\lambda + 5)(\lambda + 5) = \lambda^2 + 10\lambda + 25.$$

The characteristic polynomial in terms of l_1 and l_2 can be written as:

$$+\lambda^2 + \lambda \underbrace{\left(-2 + \frac{l_1}{2} + l_2\right)}_{=10} \underbrace{-8 + \frac{5l_1}{2} + \frac{l_2}{2}}_{=25} = 0.$$

Solving the following linear system of equations,

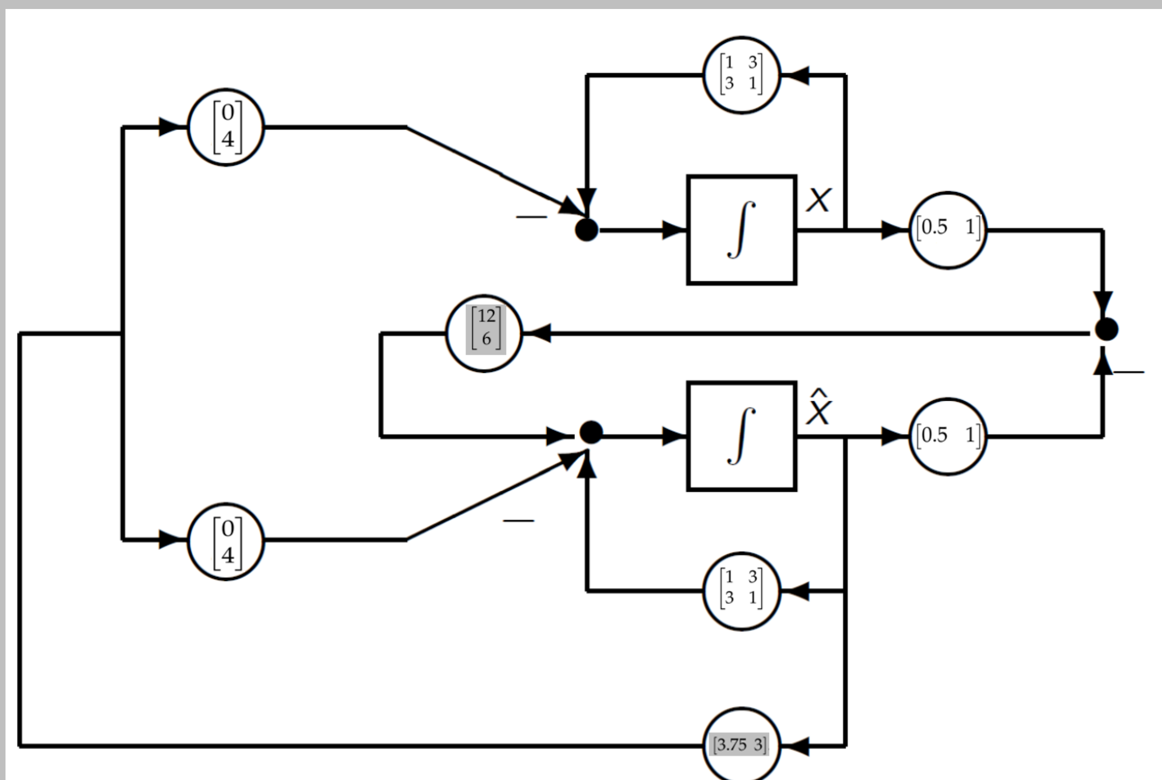
$$\begin{aligned} 25 &= -8 + \frac{5l_1}{2} + \frac{l_2}{2} \\ 10 &= -2 + \frac{l_1}{2} + l_2, \end{aligned}$$

we obtain $l_1 = 12$ and $l_2 = 6$, $\Rightarrow l = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

- Finally, overall system design:

$$\begin{aligned} u(t) &= -K\hat{x}(t) = -3.75\hat{x}_1(t) - 3\hat{x}_2(t) \\ \dot{\hat{x}}_1(t) &= \hat{x}_1(t) + 3\hat{x}_2(t) + 12(y(t) - \hat{y}(t)) \\ \dot{\hat{x}}_2(t) &= 3\hat{x}_1(t) + \hat{x}_2(t) + 4u(t) + 6(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= 0.5\hat{x}_1(t) + \hat{x}_2(t) \end{aligned}$$

(b) (5 points) Draw a block diagram representation of the overall system with the observer based controller, including the values for the gains K and L that you have designed.



4. (15 total points) Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t) \\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1. \end{aligned}$$

(a) (10 points) Find all the equilibrium points of the nonlinear system. You should find at least 2 equilibrium points.

The equilibrium points for this system are:

- $x_e^{(1)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, x_e^{(2)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

(b) (5 points) Determine the stability of the system around each equilibrium point.

The linearized system can be written as:

$$\Delta x(t) = \begin{bmatrix} x_{2e} - 2 & x_{1e} \\ 1 & -1 \end{bmatrix} \Delta x(t).$$

For the first equilibrium point:

$$\Delta x(t) = \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} \Delta x(t),$$

we obtain $eig(A) = \{1.3, -2.3\}$, hence the system is unstable around $x_e^{(1)}$. For the second equilibrium point:

$$\Delta x(t) = \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix} \Delta x(t),$$

we obtain $eig(A) = \{-3, -1\}$, hence the system is asymptotically stable around $x_e^{(2)}$.

5. (35 total points) The MPC cost function minimization can be written as:

$$\underset{\Delta U}{\text{minimize}} \quad J(\Delta U) = \frac{1}{2}(r - Wx_a - Z\Delta U)^\top Q(r - Wx_a - Z\Delta U) + \frac{1}{2}\Delta U^\top R\Delta U,$$

where $r, W, x_a, Z, Q = Q^\top \succ 0, R = R^T \succ 0$ are all constant matrices and vectors with appropriate dimensions.

(a) (20 points) Obtain the optimal ΔU^*, the optimal solution for the above unconstrained problem.

$$\frac{\partial J}{\partial \Delta U} = 0 \Rightarrow \boxed{\Delta U^* = (R + Z^\top QZ)^{-1}Z^\top Q(r - Wx_a)}.$$

We derived this in class.

(b) (5 points) Given a closed form solution for ΔU^*, show that ΔU^* is a minimizer to the above optimization problem. You have to prove that the second order necessary conditions are in fact satisfied.

The Hessian is equal to

$$R + Z^\top QZ.$$

Note that R and Z are positive positive definite matrices. Hence, you need to prove that $R + Z^\top QZ$ is also a positive definite matrix for any $Q \in \mathbb{R}^{n\times m}$. To do so, you can write $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix}$ and then expand $Z^\top QZ$ in terms of the columns of Q. Notice that R is already positive definite, and adding $q_i^\top Qq_i$ to it will also yield a positive definite matrix, and hence the Hessian is positive definite for all $q_i \in \mathbb{R}^n$. We derived this in class.

(c) (10 points) Given part (a) of this problem, suppose that you're given the following constraints on the rate of change of the control action:

$$u^{\min} \leq \Delta U \leq u^{\max}.$$

Write the corresponding optimization problem in the following form:

$$\begin{array}{ll} \text{minimize} & J(\Delta U) \\ \text{subject to} & g(\Delta U) \leq 0\,, \end{array}$$

where $g(\Delta U)$ is a linear set of constraints that you need to write as linear inequality constraints.

If $\Delta u^{\min} \leq \Delta u(k) \leq \Delta u^{\max}$, then:

$$\begin{bmatrix} -I_m \\ I_m \end{bmatrix} \Delta u(k) \leq \begin{bmatrix} -\Delta u^{\min} \\ \Delta u^{\max} \end{bmatrix}$$

For a prediction horizon N_p, we have:

$$\underbrace{\begin{bmatrix} -I_m & O & \dots & O & O \\ I_m & O & \dots & O & O \\ O & -I_m & \dots & O & O \\ O & I_m & \dots & O & O \\ \vdots & & & & \vdots \\ O & O & \dots & O & -I_m \\ O & O & \dots & O & I_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix}}_{\Delta U} \leq \underbrace{\begin{bmatrix} -\Delta u^{\min} \\ \Delta u^{\max} \\ -\Delta u^{\min} \\ \Delta u^{\max} \\ \vdots \\ -\Delta u^{\min} \\ \Delta u^{\max} \end{bmatrix}}_{b}$$

6. (25 total points) You are given the following two matrices:

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

(a) (25 points) Solve the above minimization problem for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\text{minimize} \quad x^\top Q x$$

$$\text{subject to} \quad x^\top P x = 1$$

This problem might seem difficult, but it's not. You can simply expand this problem given Q and P are derive the KKT conditions.

Given the hint, the problem becomes:

$$\begin{array}{ll} \text{minimize} & x^\top Q x \\ \text{subject to} & 1 - x^\top P x = 0. \end{array}$$

We now formulate the Lagrangian of the optimization problem:

$$\mathcal{L}(x, \lambda) = x^\top Q x + \lambda(1 - x^\top P x).$$

The first-order necessary conditions of optimality can be written produces:

$$\nabla_x \mathcal{L}(x, \lambda) = 2Qx - 2\lambda P x = 0,$$

or

$$(\lambda I_2 - P^{-1}Q)x = 0 \to \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 3/4 \end{bmatrix} \right) x \to \begin{bmatrix} \lambda - 2 & \\ & \lambda - 3/4 \end{bmatrix} x = 0.$$

Therefore, $\lambda_1 = 2, \lambda_2 = 3/4$ and the corresponding solutions are:

$$x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x^{(2)} = \pm \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

By simply comparing λ_1 and λ_2 and their corresponding solutions, we find that $\lambda_1 = 1$ is optimal in comparison with $\lambda_2 = 2$. Checking the second-order conditions, we obtain the Hessian of $\mathcal{L}(x, \lambda)$:

$$\nabla_x^2 \mathcal{L}(x^{(1)}, \lambda_1) = \begin{bmatrix} 0 & \\ & 5/4 \end{bmatrix},$$

which is positive semi-definite. Finding the tangent space and checking the positive-definiteness of the Hessian on the tangent space, we conclude that $x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a strict minimizer for the given optimization problem.

7. (10 total points) Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system. For each of the following statements, determine if it is true or false. You do *not* have to justify your answer.
   (a) (2 points) The system $x(k+1) = -Ax(k)$ is asymptotically stable.
   (b) (2 points) The system $x(k+1) = A^\top x(k)$ is asymptotically stable.
   (c) (2 points) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume A^{-1} exists).
   (d) (2 points) The system $x(k+1) = (A + A^\top)x(k)$ is asymptotically stable.
   (e) (2 points) The system $x(k+1) = A^2 x(k)$ is asymptotically stable.

(a) **True.** Eigenvalues remain in the unit disk.
(b) **True.** Eigenvalues do not change.
(c) **False.** Eigenvalues becomes larger than 1.
(d) **False.** Counter example: $A = 0.9$.
(e) **True.** If $-1 < \lambda_i < 1, \Rightarrow 0 < \lambda_i^2 < 1$.

8. (20 total points) Consider the following system

$$\begin{aligned}
\dot{x}(t) &= TJT^{-1}x(t) + Bu(t) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\
y(t) &= Cx(t) \\
&= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x(t).
\end{aligned}$$

(a) (10 points) Obtain necessary conditions on the entries of B such that only λ_1 is controllable. This means λ_2 is simply not controllable.

- First, note that $T = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$ contain the right eigenvectors and the rows of T^{-1} contain the left eigenvectors $w_{1,2,3,4}$ (the first row of T^{-1} is $w_1^\top$.)
- The left eigenvectors are provided so that you can use the eigenvector test of controllability which states that λ_1 is controllable if $w_1^\top B \neq 0$.
- For controllability of λ_1, we require:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_1 \neq 0.$$

- For λ_2 to be uncontrollable, we require:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_3 = 0.$$

- Therefore, we require $b_1 \neq 0$ and $b_3 = 0$.
- Note that since λ_1 has a single Jordan block of size 2, this implies that only one eigenvector is the solution to the eigenvector of λ_1, and that's the reason why we took w_1 as the left eigenvector and not w_2, as w_2 is a generalized evector of λ_1.

(b) (10 points) Obtain necessary conditions on the entries of C such that only λ_2 is observable. This means λ_1 is simply not observable.

- The right eigenvectors are provided so that you can use the eigenvector test for observability which states that evalue λ_i is observable if $Cv_i \neq 0$.
- For observability of λ_2, we require:

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = c_2 + c_3 \neq 0.$$

- For λ_1 to be unobservable, we require:

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = c_1 + c_4 = 0.$$

- Hence, we require $c_2 \neq c_3$ and $c_1 = -c_4$.

9. (40 total points) Solve the following miscellaneous problems.

   (a) (15 points) Determine the values of α for which the function

$$f(x,y,z) = 2x^2 + 2xz + 2\alpha yz + 2z^2$$

is concave and the values for which it is convex.

This function can be written as $f(x,y,z) = \frac{1}{2}\begin{bmatrix} x & y & z \end{bmatrix} Q \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where

$$Q = Q^\top = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 2 & 2\alpha & 4 \end{bmatrix}.$$

The leading principle minors are 4, 0, and $16\alpha^2$. Hence, for $\alpha = 0$, the Hessian matrix (or the quadratic form) is positive semidefinite, and thus the function is convex. For other values of α, the Hessian is indefinite, and hence the function is neither concave nor convex.

   (b) (15 points) Prove that the quadratic function $f(x) = x^\top Q x$ is convex if and only if $Q = Q^\top \succ 0$ (that is, Q is symmetric positive definite matrix).

This is an iff statement, so you gotta prove both directions. The first direction ($\Rightarrow$) can be proved as follows:

- Given that $f(x) = x^\top Q x$, we apply the definition of convex function.

$$\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y) \geq 0.$$

- Substituting for $f(x)$ into the LHS of the previous equation yields:

$$\alpha x^\top Q x + (1-\alpha) y^\top Q y - (\alpha x + (1-\alpha) y)^\top Q (\alpha x + (1-\alpha) y)$$

$$= \alpha(1-\alpha) x^\top Q x - 2\alpha(1-\alpha) x^\top Q y + \alpha(1-\alpha) y^\top Q y = \alpha(1-\alpha)(x-y)^\top Q (x-y)$$

- Define $z = x - y \Rightarrow$

$$\alpha(1-\alpha) z^\top Q z$$

- Since $0 \leq \alpha \leq 1$, $Q = Q^\top \succeq$ and $\forall z \Rightarrow$

$$\alpha(1-\alpha) z^\top Q z \geq 0$$

  hence, the convexity definition of a function is satisfied.

The second statement can be proved via the basic definition of convex function and positive definiteness.

(c) (10 points) Consider the following LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t).$$

where

$$A = \begin{bmatrix} 0.5a + 0.5c & 0 & 0.5a - 0.5c \\ 0 & b & 0 \\ 0.5a - 0.5c & 0 & 0.5a + 0.5c \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}.$$

This system has three eigenvalues/modes: $eig(A) = \{a, b, c\}$**.**

Which modes are **controllable**?

Mode 1. To know that, you can apply the PBH test and see that

$$\text{rank}[\lambda_i I - A \; , \; B] = \text{rank}[aI - A \; , \; B] = 3$$

whereas

$$\text{rank}[bI - A \; , \; B] < 3$$

and

$$\text{rank}[cI - A \; , \; B] < 3.$$

Hence, mode 1 defined by evalue a is the only controllable mode.

Which modes are **unobservable**?

Similar arguments to the previous problem but with applying the PBH test for observability. **Mode 3 is the only unobservable mode.**

10. (30 total points) This is a coding question.

    (a) (30 points) Write a MATLAB-based pseudo code that essentially solves a constrained MPC problem. The code should have the following **inputs**:

    - continuous time state-space matrices A, B, C defined by

      $$\dot{x} = Ax + Bu, \;\; y = Cx$$

      where matrices A, B, C are all arbitrary (i.e., of random dimensions)
    - initial conditions for the system (i.e., $x(0)$ and $u(-1) = 0$ if needed);
    - matrices Q and R defining weight matrices for the MPC problem;
    - reference signal to track r
    - input constraints on the control input given by

      $$u^{\min} \leq u(t) \leq u^{\max}$$

    - time horizon of T seconds, with a sampling time of 1 second;

    Your code should return the sequence of optimal control inputs $u(0), u(1), \ldots, u(T)$ as well as the states $x(1), \ldots, x(T+1)$. You should use the optimization method of MPC we learned in class where you construct $J(\Delta U)$ and then constrain it with the given constraint on $u(t)$. There are four important steps in your pseudo code:

    - Step 1: Obtaining discrete-time state space matrices.
    - Step 2: Finding the augmented MPC dynamics (i.e., $x_a(k)$ dynamics).
    - Step 3: Constructing matrices Z and W.
    - Step 4: Formulating the optimization problem

      $$\text{minimize} \;\; J(\Delta U) = \ldots \;\; ; \quad \text{subject to} \;\; g(\Delta U) = \ldots \leq 0$$

    - Step 5: Calling an optimization solver (you're free to use whatever you want).
    - Step 6: Extracting $u(0)$ from the solution and applying it to the system, that is, applying it to the ODE solver of $\dot{x} = Ax + Bu$.
    - Step 7: Doing the problem (Steps 1–6) in a while or a for loop until you get to the final time step. Note that some steps should not be in the for/while loop, seeing that some of them do not change (think computational efficiency). :-)

The solution of this problem can be found in the last homework.