THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 2 SOLUTION

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September 18, 2017

The objective of this homework is to test your understanding of the content of Module 2. Due date of the homework is: Friday, September 9th, 2016 @ 6:00pm. You have to upload a scanned version of your solutions on Blackboard or a typed PDF via LaTeX. If you don't have a scanner around you, you can use Cam Scanner—a mobile app that scans images in a neat way, as if they're scanned through a copier. Here's the link for Cam Scanner: https://www.camscanner.com/user/download.

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1. Go through the following links:
   (a) State space on MATLAB: http://www.mathworks.com/videos/state-space-models-part-1-creation-and-analysis-100815.html
   (b) Another way of simulating state space using ODE solver: https://www.mathworks.com/matlabcentral/answers/146782-solve-state-space-equation-by-ode45

2. In this link http://academic.csuohio.edu/richter_h/courses/mce371/mce371_5.pdf, you'll find a quick introduction to state space and its implementation on MATLAB, similar to the one above.
   (a) Go through Pages 3–14 of this PDF presentation. Make sure that you understand the details involved.
   (b) You are now given the following dynamical system (identical to the one given in the PDF):
   $$2y^{(4)}(t) + 0.9y^{(3)}(t) + 45.1\ddot{y}(t) + 10\dot{y}(t) + 250y(t) = 250u(t),$$
   where $y(t)$ and $u(t)$ are the output and input to the system. Derive **two different** state space representations, i.e., **obtain two sets** of state-space matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ for this fourth order ODE.
   Two different state space realizations can be the controllable and observable canonical forms, which were given by the forms derived in Module 02.
   (c) For each set of the derived matrices, and given what you learned about the ODE solvers on MATLAB, simulate the dynamics of this system assuming that the input $u(t)$ is a unit step function ($u(t) = 1$). Consider that the time horizon is equal to 2 seconds. You'll have to plot the states of the system with respect to time, as well as the output $y(t)$. You can assume any set of initial conditions (do not change the initial conditions for the two state space representations).
   Solution: You can obtain the solutions as instructed in the link above.
   (d) Is there a difference between the output and the states for the two state-space representations? Why/Why Not? Explain your answer.
   (e) Find the transfer function associated to the two distinct state space representations that you derived in 2-(b). Is the transfer function unique? Why/Why Not?

3. Assume that two systems, N_1 and N_2, are cascaded in series. System N_1 is defined by the derivative operator (i.e., the output to N_1 is the derivative of its input), and system N_2 is defined by a function $\gamma(t)$ (i.e., the output of N_2 is the input of N_2 multiplied by $\gamma(t)$). Is the overall, cascaded system linear? Nonlinear? Time-varying? Time-invariant? Causal? Prove it.

   Solution: First, note that $y(t) = N(u(t)) = N_2(N_1(u(t))) = \gamma(t)\dot{u}(t)$. This system is linear, time-varying, causal, and lumped. The system is linear as
   $$N(\alpha_1 u_2 + \alpha_2 u_2) = N_2(N_1(\alpha_1 u_2 + \alpha_2 u_2)) = \gamma(t)(\alpha_1 \dot{u}_1(t) + \alpha_2 \dot{u}_2(t)) = \alpha_1 N(u_1(t)) + \alpha_2 N(u_2(t)).$$
   Clearly the system is time-varying as the term $\gamma(t)$ makes it so. If the input is shifted by T, i.e., $u(t) := u(t-T)$ then $y(t-T) = \gamma(t-T)\dot{u}(t-T) \neq \gamma(t)\dot{u}(t-T)$, hence the system is time-varying.

4. What happens when two systems, out of which one is LTI while the other is LTV, are connected together in series? Would the cascaded overall system still remain linear?

   Now suppose that the order of the cascading is reversed, does that change the overall system output given the same input? Yes/No answers do not suffice. You have to prove your result.

   Solution: Let N_1 and N_2 be the two LTI and LTV systems. Then, $y(t) = N(u(t)) = N_2(N_1(u(t)))$. To test for linearly, assume that $u(t) = \alpha_1 u_1(t) + \alpha_2(t) u_2(t)$ is the input to the system. Then:

$$y(t) = N_2(N_1(\alpha_1 u_1(t) + \alpha_2(t) u_2(t))) = N_2(\alpha_1 N_1(u_1(t)) + \alpha_2 N_1(u_2(t)))$$

$$= \alpha_1 N_2(N_1(u_1(t))) + \alpha_2 N_2(N_1(u_2(t))) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 N(u_1(t)) + \alpha_2 N(u_2(t)).$$

   Therefore, the cascaded system is in fact linear. The order does not matter, but the output will be different.

5. Assume that a system N is linear, with an output defined as $y(t) = N(u(t))$. Prove that if the input to the system is zero for all $t \geq 0$, then the output must be also 0 for all $t \geq 0$.

   Solution: Since the system N is defined to be linear, then $N(u(t) = 0) = N(0 \times u(t) = 0) = 0 \times N(u(t) = 0) = 0(t) = 0$. Hence, the always zero input for linear system gives the always zero output.

6. A Trump-Clinton dynamical system that exists nowhere follows these two differential equations:

$$\begin{aligned} \ddot{T}(t) + \alpha_1(t)\dot{T}(t) - \alpha_2(t)\dot{C}(t) &= \alpha_3(t)u(t) & (1) \\ \dot{C}(t) &= \alpha_4(t)u(t) - C(t) - \alpha_5(t)T(t), & (2) \end{aligned}$$

   where $T(t)$ and $C(t)$ are the two mental **states** of Jalyooka Trump and Palyooka Clinton, $u(t)$ is the control input, and $\alpha_i(t)$ functions are all time-varying functions. Derive the state-space representation of this non-existent dynamical system. You should be able to obtain an equation similar to this:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}(t)\mathbf{x}(t) + \boldsymbol{B}(t)u(t),$$

   where $\mathbf{x}(t)$ is the state-vector of the system (minimum of size 3) and $\boldsymbol{A}, \boldsymbol{B}$ are state-space matrices that you should derive (in terms of $\alpha(t)$ functions).

   Hint: let $x_1(t) = T(t)$ an $x_3(t) = C(t)$.

   Solution: Choosing state variables as $x_1(t) = T(t), x_2(t) = \dot{T}(t)$ and $x_3(t) = C(t)$, we obtain:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha_2(t)\alpha_5(t) & -\alpha_1(t) & -\alpha_2(t) \\ -\alpha_5(t) & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_3(t) + \alpha_2(t)\alpha_4(t) \\ \alpha_4(t) \end{bmatrix} u(t) \Rightarrow$$

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}(t)\mathbf{x}(t) + \boldsymbol{B}(t)u(t),$$

7. The dynamics of the vertical ascent of a certain rocket (that does not blow up, Wink Wink SpaceX ;)) can be modeled as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g\left(\dfrac{D}{x_1(t) + D}\right)^2 + \dfrac{\ln(u)}{m} \end{bmatrix},$$

   where D is the distance from earth to the surface of the rocket (assumed to be constant), m is the actual mass of the rocket, g is the gravity constant, and u is the thrust that is assumed to be constant.

   Find the equilibrium states (x_1^*, x_2^*) of the above dynamic system.

   Solution: Setting $\dot{x}_1(t) = \dot{x}_2(t) = 0$, we obtain $x_2^* = 0$ and $x_1^* = \sqrt{\dfrac{mgD^2}{\ln(u)}} - D$