THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 3 SOLUTION

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The objective of this homework is to test your understanding of the content of Module 3. Due date of the homework is: Friday, September 23rd, 2016 @ 11:59pm. You have to upload a scanned version of your solutions on Blackboard or a typed PDF via LaTeX.

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1. Determine which of the following sets are vector spaces. Prove your answer.

   (a) The set of natural numbers.

   Solution: The set of natural numbers ($\mathbb{N}$) is NOT a vector space, as it does not satisfy the multiplication by a scalar property, i.e., if you multiply $x \in \mathbb{N}$ by $\alpha = 0.5 \in \mathbb{R}$, you'll get $0.5x \in \mathbb{R} \notin \mathbb{N}$.

   (b) The set of square diagonal matrices of size n.

   Solution: Clearly, the set of diagonal matrices is a vector space—if you add two diagonal matrices, you'll obtain a diagonal matrix. If you scale a diagonal matrix, you'll obtain—yip, you guessed it right—a diagonal matrix[1]. Here, we assume that the size of the matrix is fixed.

   (c) The set of (square) strictly upper diagonal matrices ($a_{i,j} = 0$ for $i \geq j$).

   Solution: This set is indeed a vector space. Similar argument to the previous question. Note that you can also find the dimension to this vector space. Since we're assuming that we have upper diagonal matrix, then we have at most $n(n-1)/2$ distinct elements, which means your basis for this vector space are of dimension of $n(n-1)/2$.

   (d) The set of bounded sequences, i.e., $\{u[k], k = 0, 1, \ldots, ; |u(k)| < \infty\}$.

   Solution: This set is a vector space, as the sum of two bounded sequences is always bounded. Since the bound is not specified (i.e., it's infinity), this is a vector space. Note that if the sequence was said to be bounded, then this space won't be a vector space.

   (e) The set of bounded functions $u(t)$ on a predefined interval, such that $|u(t)| \leq K$, where K is a positive number.

   Solution: Not a vector space. You can think of a simple counter example.

2. Find the null space, range space, determinant, and rank of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}.$$

   Solution:

   (a) $\text{Null}(A) = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\text{R}(A) = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\det(A) = 0$, $\text{rank}(A) = 2$.

   (b) $\text{Null}(B) = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\text{R}(B) = \text{span}(I_3)$, $\text{rank}(B) = 3$.

   (c) $\text{Null}(C) = \alpha_1 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\text{R}(C) = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\text{rank}(C) = 2$.

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[1] You don't always have to use symbols and equations to tell me what you mean; you just have to be abstract and concise.

3. Assume that $A = TDT^{-1}$, where D is the diagonal matrix.
   (a) Prove by mathematical induction that $A^k = TD^kT^{-1}$.
   Solution: First, set $k = 1$, then $A^1 = TD^1T^{-1} = TDT^{-1}$. Now, assume that the given equation is true for k, prove the result for $k+1$:
   $$A^{k+1} = A^kA = TD^kT^{-1}A = TD^kT^{-1}TDT^{-1} = TD^{k+1}T^{-1}.$$
   (b) Prove that $e^{At} = Te^{Dt}T^{-1}$.
   $$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = \sum_{i=0}^{\infty} \frac{(TDtT^{-1})^i}{i!} = \sum_{i=0}^{\infty} T\frac{(Dt)^i}{i!}T^{-1} = T\left(\sum_{i=0}^{\infty} \frac{(Dt)^i}{i!}\right)T^{-1} = Te^{Dt}T^{-1}.$$

4. For the following dynamical system:
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$.

Solution: First, find the exponential of the matrix A: $e^{At} = \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}$, since A is nilpotent of order 2, hence $e^{At} = I + At$. Hence, let $x(2)$ be your $x(t_0)$, thus:
$$x(t) = e^{A(t-t_0)}x(t_0) = \begin{bmatrix} 1 & 0 \\ 2(t-2) & 1 \end{bmatrix} x(2) \Rightarrow x(0) = \begin{bmatrix} 1 & 0 \\ 2(0-2) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

5. For the same dynamical system in the previous problem, find $x(0)$ when $u(t) = 1$ and $x(2)$ is the zero vector.

Solution: Similar to the above problem,
$$x(0) = \begin{bmatrix} 1 & 0 \\ 2(0-2) & 1 \end{bmatrix} x(2) + \int_2^0 \begin{bmatrix} 1 & 0 \\ 2(0-\tau) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau = 0 + \int_0^2 \begin{bmatrix} 1 \\ -2\tau \end{bmatrix} d\tau = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

6. You are given that $A = \begin{bmatrix} A_1 & I \\ 0 & A_1 \end{bmatrix}$ where A_1 is a square matrix of dimension n, and A is a square matrix of dimension $2n$.
   (a) Find e^{At} in the simplest possible form.
   Solution: First, note that $A = \begin{bmatrix} A_1 & I \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = X + Y$
   Notice that the two matrices X, Y defined above commute, as $XY = YX = \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix}$. Hence, we can use the hint to obtain $e^{At} = e^{Xt}e^{Yt}$. Since X is a block diagonal matrix, then $e^{Xt} = \begin{bmatrix} e^{A_1t} & 0 \\ 0 & e^{A_1t} \end{bmatrix}$. Note that matrix Y is nilpotent of order 2, then $e^{Yt} = I + Yt = \begin{bmatrix} I & It \\ 0 & I \end{bmatrix}$. Hence:
   $$e^{At} = \begin{bmatrix} e^{A_1t} & 0 \\ 0 & e^{A_1t} \end{bmatrix} \begin{bmatrix} I & It \\ 0 & I \end{bmatrix} = \begin{bmatrix} e^{A_1t} & te^{A_1t} \\ 0 & e^{A_1t} \end{bmatrix}$$
   (b) Assume now that $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Find e^{At}.
   Solution:
   $$e^{A_1t} = A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ & e^{\alpha t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{A_1t} & te^{A_1t} \\ 0 & e^{A_1t} \end{bmatrix} = \text{you can fill in the blanks}$$

7. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t).$$

   (a) Find $e^{A(t-t_0)}$.
   Solution: This matrix is nilpotent of order 3, i.e., $A^3 = 0$. Hence,

$$e^{A(t-t_0)} = I + A(t-t_0) + A^2(t-t_0)^2/2 = \begin{bmatrix} 1 & 0 & 0 \\ 2(t-t_0) & 1 & 0 \\ 6(t-t_0)^2 & 6(t-t_0) & 1 \end{bmatrix}.$$

   (b) Given that $x(1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, compute $x(t)$ for $t \geq 1$.
   Solution: Assume that $t_0 = 1$, hence,

$$x(t) = e^{A(t-1)}x(1) = \begin{bmatrix} 1 \\ 2t-1 \\ 6t^2 - 6t + 1 \end{bmatrix}.$$

   (c) What is $x(5)$?
   Solution: $x(5) = \begin{bmatrix} 1 \\ 9 \\ 121 \end{bmatrix}$.

   (d) Now assume that $x(1) = 0$, and the control input is $u(t) = 1$. Find the initial condition $x(0)$ that would lead to $x(1)$. In other words, assume that your initial condition is now $x(0)$, which you're required to find given that the control drives the system back to zero.
   Solution:

$$x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2(t-t_0) & 1 & 0 \\ 6(t-t_0)^2 & 6(t-t_0) & 1 \end{bmatrix} x(t_0) + \int_{t_0}^{t} \begin{bmatrix} 1 \\ 2(t-\tau) \\ 6(t-\tau)^2 \end{bmatrix} u(\tau)d\tau.$$

   Hence, and given that $x(1) = 0$, then

$$x(0) = \int_1^0 \begin{bmatrix} 1 \\ -2\tau \\ -6\tau^2 \end{bmatrix} d\tau = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

8. Find e^{At} for the following matrices. The expression you obtain should be a closed form one.

   (a) $A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$

   Solution: $e^{At} = \begin{bmatrix} 1+at & -at \\ at & 1-at \end{bmatrix}$ as A is nilpotent of order 2.

   (b) $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, a+b+c = 0$

   Solution: Note that A A is nilpotent of order 2 if $a+b+c=0$, hence:

$$e^{At} = I + At = \begin{bmatrix} 1+at & bt & ct \\ at & 1+bt & ct \\ at & bt & 1+ct \end{bmatrix}.$$

   (c) $A = \lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$

   Solution: this matrix is scaled by λ_1, hence:

$$e^{At} = \begin{bmatrix} 1+\lambda_1 at & -\lambda_1 at \\ \lambda_1 at & 1-\lambda_1 at \end{bmatrix}$$

(d) $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$

Solution: This is a Jordan block in it's most boring form. You know how to do this.

You can confirm your answers on MATLAB. Show your code.

9. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \left(\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} + \lambda I_3 \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $a + b + c = 0$. Find $x(0)$ if $u(t) = 2e^{\lambda t}, \forall t \geq 0$, and $x(2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$.

Solution: First, the matrix λI_3 computes with $\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Hence, we can write:

$$e^{At} = e^{\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} t} \cdot e^{\lambda I t} = e^{\lambda t} \begin{bmatrix} 1 + at & bt & ct \\ at & 1 + bt & ct \\ at & bt & 1 + ct \end{bmatrix}.$$

Now that we have the matrix exponential, we can find $x(0)$ via this equation:

$$x(0) = e^{-2A} x(2) + \int_2^0 e^{-A\tau} B u(\tau) d\tau = e^{-2\lambda} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

10. Prove the following results:

(a) If $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{bmatrix}$.

Solution:

$$e^{At} = \sum_{i=0}^{\infty} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}^i \frac{t^i}{i!} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} t + \frac{t^2}{2!} \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} + \frac{t^3}{3!} \begin{bmatrix} 0 & -a^3 \\ a^3 & 0 \end{bmatrix} = \begin{bmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{bmatrix},$$

since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

(b) If $A = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}$.

Solution: Similar approach to the previous problem. Note that

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(c) If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, then $e^{At} = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$.

Solution: Note that $A = aI + \begin{bmatrix} & b \\ -b & \end{bmatrix} = X + Y$, and matrices X, Y commute. Hence, $e^{At} = e^{at}e^{Yt}$, where e^{Yt} is the matrix from part (a) above.

11. Find the generalized eigenvectors for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, the Jordan canonical form, as well as the matrix exponential e^{At}.

Solution: By finding the evalues of A, we get three evalues at $\lambda = 1$. To find the number of Jordan blocks associated with this evalue, we should find the geometric multiplicity of this evalue with algebraic multiplicity of 3. We solve this equation for the evectors of λ :

$$(A - I)v = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} v = 0.$$

The only solution of this linear system of equations is $v_1^1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Hence, there's only one Jordan block associated to this matrix as the geometric multiplicity is equal to 1. Next, we find the generalized evalues v_1^2, v_1^3. We can solve:

$$(A - I)v_1^2 = v_1^1, \ (A - I)v_1^3 = v_1^2.$$

This yields:

$$v_1^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_1^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$A = TJT^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1}$$

$$\Rightarrow e^{At} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t & 0.5t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1}$$