HOMEWORK # 4

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The objective of this homework is to test your understanding of the content of Module 4. Due date of the homework is: Thursday, October 5th @ 11:59pm.

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1. Consider the following time-varying system:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} x(t).$$

   (a) Find the state transition matrix.

   (b) Now assume that the system starts from an unknown $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, where $x_1(0) = 1$. Also, assume that $x_2(t = 4) = -1$. Given this, find $x_2(0)$.

   (c) After obtaining $x_2(0)$ and given $x_1(0) = 1$, find a general from of $x(t)$ starting $x(0)$ as initial conditions.

   (d) What happens to $x_1(t)$ and $x_2(t)$ as $t \to \infty$?

2. The STM of $\dot{x}(t) = A(t)x(t)$ is given as follows:

$$\phi(t, t_0) = e^{t_0^2 - t^2} \begin{bmatrix} 1 & \ln\left(\frac{t+1}{t_0+1}\right) \\ 0 & 1 \end{bmatrix}.$$

   (a) Given any STM, how can you obtain $A(t)$ back? Prove that $A(t) = \dot{\phi}(t, t)$ for any LTV system.

   (b) For the STM given in this problem, obtain $A(t)$.

   (c) Find $\phi^{-1}(t, t_0)$

3. The following system

$$\dot{x}(t) = A(t)x(t) + e^{-\alpha^2(t)} \begin{bmatrix} \pi \\ 0 \end{bmatrix} u(t).$$

   The STM for this system is given by:

$$\phi(t, t_0) = e^{\alpha^2(t_0) - \alpha^2(t)} \begin{bmatrix} 1 & 0 \\ \cos(\pi t) - \cos(\pi t_0) & 1 \end{bmatrix}.$$

   (a) Compute $A(t)$.

   (b) Determine the inverse of the STM.

   (c) Find $x(t)$ if $x(t_0) = 0$ and $u(t) = 1$.

4. Find the STM associated with

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

5. Consider the following dynamical system:

$$\dot{x}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} x(t).$$

(a) Derive this structure for the STM:

$$\phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ 0 & \phi_{22}(t,t_0) \end{bmatrix}.$$

You should basically show that $\dot{\phi}_{ii}(t,t_0) = A_{ii}(t)\phi_{ii}(t,t_0)$, and then show a explicit form for $\phi_{12}(t,t_0)$.
Hint: Remember that $\dot{phi}(t,t_0) = A(t)\phi(t,t_0)$ and $\phi(t_0,t_0) = I$. You should use that to prove the following result:

$$\phi_{12}(t,t_0) = \int_{t_0}^{t} \phi_{11}(t,\tau)A_{12}(\tau)\phi_{22}(\tau,t_0)d\tau.$$

(b) Assume that the dynamics for a system are given by:

$$\dot{x}(t) = \begin{bmatrix} 0 & 2t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2t \end{bmatrix} u(t).$$

Use the results you developed in part (a) to determine $x(t)$ giving that the initial conditions for the system are zero, and $u(t) = 1$, without evaluating this integral $\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau$.

6. Building on the theoretical results from Problem 5, find the STM for

$$A(t) = \begin{bmatrix} -2 & 0 & t \\ 2t & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

You should use the result you proved in Problem 5:

$$\phi_{12}(t,t_0) = \int_{t_0}^{t} \phi_{11}(t,\tau)A_{12}(\tau)\phi_{22}(\tau,t_0)d\tau.$$

7. You are given the following system:

$$\dot{x}(t) = \left(\dot{\lambda}(t)I + \begin{bmatrix} a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \end{bmatrix}^{\bullet} \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $\left[M(t)\right]^{\bullet}$ denotes the derivative of the matrix, i.e., derivative of the individual entries of the matrix.

(a) Assume that $a(t) + b(t) + c(t) = 0$ for all t. Find a simple expression for $\phi(t,t_0)$.

(b) Given that the control input is $u(t) = 2e^{\lambda(t)}$, and that $x(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, obtain $x(0)$.