HOMEWORK # 5

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Due date of the homework is: Friday, October 13th @ 11:59pm.

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1. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

   (a) Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.

   (b) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}1^+(k)$ and $x(0) = 0$.

2. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{D} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

   (a) Find a general expression for D^k.

   (b) Find A^k.

   (c) Compute $x(k)$ if the control input is null.

   (d) Computer $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k 1^+(k)$ and $\lambda_1 = 4$.

3. This problem requires you to think deeply about the problem and to remember the linear algebra background we discussed in Module 2. Consider the following system with two inputs $\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = u(k)$ and the following dynamics:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u(k), x(0) = 0.$$

   (a) By setting $u_2(k) = 0 \ \forall k$, and using $u_1(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_1(k)$ that would achieve that for $k = 0, 1, 2$.

   (b) By setting $u_1(k) = 0 \ \forall k$, and using $u_2(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_2(k)$ that would achieve that for $k = 0, 1, 2$.

   (c) Assume at $k = 0, 1$, only u_1 can be used and at $k = 2$, only u_2 can be used. Find the input $u(k) \forall k$ such that the state can be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

4. You are given this system:

$$x(k+1) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), a \neq 0, b \neq 0.$$

(a) Prove that $A^k = \begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix}$.

(b) If $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u(k) = 0$, find $x(0)$.

(c) Find $x(k)$ if $u(k) = a^k$ and $x(0) = 0$.

5. You're given the following DT LTV system:

$$x(k+1) = A(k)x(k) + B(k)u(k).$$

(a) Derive a system of equations whose solution gives the three inputs $u(0), u(1)$ that would drive the system from state $x(0)$ to $x(2)$.

(b) Now assume that

$$A(k) = \begin{bmatrix} 0 & 2-k \\ 0 & 0 \end{bmatrix}, B(k) = \begin{bmatrix} 2-k & 0 \\ 0 & 2-k \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the input sequence $u(0), u(1)$ that would steer the system from $x(0)$ to $x(2)$.

6. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1) \\ \dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3 \end{aligned}$$

(a) Find all the equilibrium points of the nonlinear system.

(b) Determine the stability of the system around each equilibrium point, if possible.

(c) Solve the same problem if the system is in discrete time:

$$\begin{aligned} x_1(k+1) &= x_2(k)(x_1^2(k) - 1) \\ x_2(k+1) &= x_2^2(k) + x_1(k) - 3. \end{aligned}$$