THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 6
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Due date of the homework is: Wednesday, November 8th @ 11:59pm.

1. Prove that the system represented in the controllable canonical form is always controllable.

2. Show that the controller design

$$u(t) = -B^\top e^{A^\top (t_f - t)} W^{-1}(t_f) \left[ e^{At_f} x_0 - x_{t_f} \right]$$

   steers the system from $x(t_0) = x_0$ to $x(t_f) = x_{t_f}$.

3. You are given the following CT LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(t).$$

   We wish to find a state feedback controller $u = Kx$ (not $u = -Kx$) such that $A_{cl} = A + BK$ is **block diagonal** with eigenvalues $\lambda_{1,2} = \{2, 3\}$ assigned to the first diagonal block, and eigenvalues $\lambda_{3,4} = \{0, 1\}$ assigned to second diagonal block. Note that your $K$ matrix can be written as: $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$.

4. Answer the following questions for this system:

$$x(k+1) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k).$$

   (a) Is the system controllable? Show this result via the first three controllability tests. You can use MATLAB in this problem to find the eigenvalues and the eigenvectors.

   (b) Is the system stabilizable? If so, design a state feedback controller $u(k) = -Kx(k)$ that would shift the unstable eigenvalues to stable locations which are $\lambda_{cl}(A) = \{-1, -0.5, 0.5\}$. Can you obtain such a state feedback controller? Here $K \in \mathbb{R}^{1\times 3}$.

   (c) Consider that $x(0) = 0$. Obtain the reachable subspace $\mathcal{R}_k$ of the system at $k = 1, 2, 3, \ldots$. Recall that the reachable subspace is

$$\mathcal{R}_k = \text{Range-Space}(\begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix}).$$

   (d) Can you find a control sequence $(u(0), u(1), \ldots, u(n-1))$ that can drive the system from $x(0) = 0$ to $x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in the least possible time-steps $n$. You can start by trying $n = 1$ then $n = 2$, etc...

5. Show that if we discretize a system $\dot{x}(t) = Ax(t) + Bu(t)$ to $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$, then if the discretized system defined by $(\tilde{A}, \tilde{B})$ is controllable, then so should the continuous system defined by the pair $(A, B)$.

   *Hint:* You can prove this result by contradiction and by using the eigenvector test for controllability.

6. For a CT LTI system with

$$A = \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 2 & 1 \\ & & & & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ \alpha_1 & -1 \\ 1 & \alpha_2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

obtain condition(s) on $\alpha_1$ and $\alpha_2$ such that the pair $(A, B)$ is controllable. You can use any test that you want, but the choice of the test might incur longer time to come to the condition—choose wisely!

7. Consider the following system

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u(t).$$

   (a) Is the system controllable?

   (b) Find the controllability subspace.

   (c) Is the system stabilizable?

   (d) Is there a state feedback gain such that $A - BK$ has evalues $\{-1, 1\}$?

   (e) Suppose that $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $u(t) = 0$. Find $y(t)$ if $y(t) = Cx(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(t)$.

8. Coding problem—horray! Ok, so in this problem I want you to write MATLAB function that takes as an input (potentially large) matrices $A$ and $B$ and checks for the four tests of controllability (T1—T4) we learned in class. You can assume that the system is continuous time. Some instructions:

   (a) Implement the four tests as discussed in class in the most efficient way.

   (b) The function should return a 4-digit binary string with $\{1, 1, 1, 1\}$ denoting that the system is controllable and the four tests all yielded a '1' meaning the system is controllable or $\{0, 0, 0, 0\}$ denoting the system is not controllable. If your code returns something other than that, then you've probably done something wrong.

   (c) The function should also return the computational time required to run each of the methods (T1—T4). You should start by trying your code for random small-scale dynamic systems, and then try a rather large-scale system with millions of states and tens of thousands of control inputs. The computational time should be returned in an array with four numbers—each number denoting how many seconds it took to run each test. See MATLAB's help files to compute computational time for any function.

   (d) Your MATLAB function essentially has four other functions that you call for each test. You can assume that $t_f = 10$ for Test 4 (the Gramian test).

   (e) I'll give extra credits how try hard and do a good job.

   (f) Please upload: (a) a PDF for the solutions of the homework, and (b) a single m-file with the code that returns the above outputs. Not two PDFs and four m-files, a single PDF and a single m-file—please. :)