THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 6 SOLUTIONS

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Due date of the homework is: Tuesday, November 21st, 2016 @ 11:59pm. You have to upload a **clear** scanned version of your solutions on Blackboard or a typed PDF via LaTeX.

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1. Prove that the system represented in the controllable canonical form is always controllable.

2. Show that the controller design

$$u(t) = -B^\top e^{A^\top (t_f - t)} W^{-1}(t_f) \left[e^{At_f} x_0 - x_{t_f} \right]$$

steers the system from $x(t_0) = x_0$ to $x(t_f) = x_{t_f}$.

3. You are given the following CT LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(t).$$

We wish to find a state feedback controller $u = Kx$ (not $u = -Kx$) such that $A_{cl} = A + BK$ is **block diagonal** with eigenvalues $\lambda_{1,2} = \{2, 3\}$ assigned to the first diagonal block, and eigenvalues $\lambda_{3,4} = \{0, 1\}$ assigned to second diagonal block. Note that your K matrix can be written as: $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$.

Solution: Since $u = Kx$, we can write $A + BK$ as:

$$A + BK = \begin{bmatrix} k_1 & 1+k_2 & 1+k_3 & 1+k_4 \\ 1 & 0 & 0 & 0 \\ 1+k_5 & 1+k_6 & k_7 & k_8 \\ 1+k_5 & 1+k_6 & 1+k_7 & k_8 \end{bmatrix}.$$

Since we want $A + BK$ to be block diagonal, then the off-diagonal blocks must be all zeros. Hence,

$$1 + k_3 = 1 + k_4 = 1 + k_5 = 1 + k_6 = 0 \implies k_3 = k_4 = k_5 = k_6 = -1.$$

Since the first diagonal block is assigned evalues 2 and 3, we can write:

$$(\lambda - 2)(\lambda - 3) = \lambda^2 - 5\lambda + 6 \equiv \lambda^2 - k_1\lambda - (1 + k_2) \implies k_1 = 5, k_2 = -7.$$

Similarly, we obtain

$$\lambda(\lambda - 1) = \lambda^2 - \lambda + 0 \equiv (\lambda - k_7)(\lambda - k_8) - k_8(1 + k_7) \implies k_7 = 1, k_8 = 0.$$

Hence, the gain matrix K can be written as:

$$K = \begin{bmatrix} 5 & -7 & -1 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

4. Answer the following questions for this system:

$$x(k+1) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k).$$

(a) Is the system controllable? Show this result via the first three controllability tests. You can use MATLAB in this problem to find the eigenvalues and the eigenvectors.

Solution:

[T1] The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

which is full rank. Hence, the system is stabilizable.

[T2] The evalues of A are: $\{-1, 1, 2\}$. Using the PBH test, we find:

$$\lambda_1 = -1, \quad \text{rank}[A - \lambda_1 I\ ,\ B] = \text{rank} \begin{bmatrix} -2 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} = 3 \Rightarrow \lambda_1 \text{ passes PBH test}$$

For other evalues, check best student solutions.

[T3]

(b) Is the system stabilizable? If so, design a state feedback controller $u(k) = -Kx(k)$ that would shift the unstable eigenvalues to stable locations which are $\lambda_{cl}(A) = \{-1, -0.5, 0.5\}$. Can you obtain such a state feedback controller? Here $K \in \mathbb{R}^{1\times 3}$.

Solution: Since the system is controllable, then it is also stabilizable (by definition). We can obtain $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ by matching the characteristic polynomial of $A - BK$ with the characteristic polynomial of $\lambda_{cl}(A) = \{-1, -0.5, 0.5\}$:

$$(\lambda + 1)(\lambda + 0.5)(\lambda - 0.5) = \lambda^3 + \lambda^2 - 0.25\lambda - 0.25 = 0 \equiv$$

$$\lambda^3 + (k_3 - 2)\lambda^2 + (k_1 + k_2 - 1)\lambda + k_1 - 2k_3 - k_3 + 2 = 0.$$

Hence, $k_3 = 3, k_2 = 0, k_1 = 0.75$.

(c) Consider that $x(0) = 0$. Obtain the reachable subspace $\mathcal{R}_k$ of the system at $k = 1, 2, 3, \ldots$. Recall that the reachable subspace is

$$\mathcal{R}_k = \text{Range-Space}(\begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix}).$$

Solution: The reachable subspace is

$$\mathcal{R}_k = \text{Range-Space}(\begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix}),$$

hence:

- $k = 1$, $\mathcal{R}_1 = \text{Range-Space}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.
- $k = 2$, $\mathcal{R}_2 = \text{Range-Space}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\right)$.
- $k = 3$, $\mathcal{R}_3 = \mathcal{R}_4 = \mathcal{R}_5 = \cdots = \mathbb{R}^n$ since the system is controllable.

(d) Can you find a control sequence $(u(0), u(1), \ldots, u(n-1))$ that can drive the system from $x(0) = 0$ to $x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in the least possible time-steps n. You can start by trying $n = 1$ then $n = 2$, etc...

Solution: Notice that $x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a vector that spans $\mathcal{R}_2$. To see that, you can write $x(n)$ as a linear combination of the columns of $\mathcal{R}_2$. You can also solve this equation for $u(0), u(1)$:

$$x(2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \Rightarrow u(0) = 1, u(1) = -2.$$

Hence, the least possible time-steps is $n = 2$—you cannot get to $x(n)$ in one step (you can also get to $x(n)$ is $n \geq 2$ time-steps).

5. Show that if we discretize a system $\dot{x}(t) = Ax(t) + Bu(t)$ to $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$, then if the discretized system defined by $(\tilde{A}, \tilde{B})$ is controllable, then so should the continuous system defined by the pair (A, B).

   Hint: You can prove this result by contradiction and by using the eigenvector test for controllability.

6. For a CT LTI system with

$$A = \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 2 & 1 \\ & & & & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ \alpha_1 & -1 \\ 1 & \alpha_2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

   obtain condition(s) on α_1 and α_2 such that the pair (A, B) is controllable. You can use any test that you want, but the choice of the test might incur longer time to come to the condition—choose wisely!

   Solution: Notice that the evalues for A are $\{-1, -1, -1, -1, 2, 2\}$. Hence, A has two distinct evalues. Given that, the easiest way to solve this problem would be to use the PBH test. The only condition that is generated for the PBH that makes the two evalues pass the test is $\boxed{\alpha_1\alpha_2 \neq -1}$. If the product of α_1 and α_2 is equal to -1, then system would fail the PBH test, and hence (A, B) would not be controllable.