THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 8
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Due date of the homework is: April 16th, @ 11:59pm.

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1. The following LTI system is given:

$$\dot{x}(t) = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

   (a) Assume that $\alpha = 2$. Find a state feedback controller $u(t) = -Kx(t)$ such that the closed loop evalues are both at -1.

   (b) Assume now that α is not known. Find the range of values of α that would make the state-feedback controlled system stable. That is, set $u(t) = -Kx(t)$ and find the range of values of α that would produce a stable closed loop system performance.

   (c) For the question in part (b), what is the percentage of change in α from $\alpha = 2$ that we considered in part (a), that the system can tolerate before becoming unstable?

   (d) Assume again that $\alpha = 2$, and that $x(t)$ is not available in real-time. This requires building a state estimator or an observer. Design an observer gain L such that the estimation error evalues are both at -1.

   (e) After solving (d), combine the controller from (a) with the observer in (d) to arrive at an observer-based state-feedback controller.

   (f) Implement the whole setup on MATLAB via the `ode45` solver. Show that the observer is succeeding in estimating the system states. Consider that the estimator initial conditions are zero (i.e., $\hat{x}(0) = 0$) and that the system's initial conditions are $x(0) = \begin{bmatrix} -2 & 2 \end{bmatrix}^\top$.

   (g) Assume now that we connect the observer designed in part (d) with the system with unknown α in part (b). What are the values of α that would drive the observer-based controller system to stay stable?
   Hint: For any fourth order polynomial

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

   to have roots with negative real parts, the necessary and sufficient conditions are that:

$$\{a_3, a_2, a_1, a_0\} > 0, \quad a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 > 0.$$

2. The following autonomous LTI system is given:

$$\dot{x}(t) = Ax(t) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x(t), \quad y(t) = Cx(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

   (a) Is the above system observable? Prove that via the four tests of observability.

   (b) What is the unobservable subspace (if any)?

   (c) Is the above system detectable?

   (d) Design a Luenberger observer such that the closed loop estimation error dynamics have both evalues placed at -1.

   (e) Show an overall diagram of the dynamic system and the observer.

   (f) Find $\lim_{t\to\infty} e(t)$?

(g) Assume now we slightly change the dynamics of the system. Specifically, we change the A-matrix to:

$$\bar{A} = \begin{bmatrix} -0.9 & 1 \\ 0 & 1 \end{bmatrix}.$$

Given this slight change in matrix A, will the observer designed in the previous part yield converging and asymptotically stable estimation error dynamics or no?

Hint: Derive the closed loop dynamics of $\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix}$ and investigate the stability of the closed loop estimation error dynamics.

3. You are given a CT-LTI system defined by

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) Design a dynamic observer such that the error dynamics have eigenvalues at -4 and -2. First, you should derive the dynamics of the observer and explain how it works. Then, obtain a gain L that achieves the desired objective.

(b) Finally, plot the norm of the estimation error dynamics, i.e., $||e(t)||_2$ as a function of time given that the observer's initial conditions are $\hat{x}(0) = \begin{bmatrix} 10 & -10 \end{bmatrix}^\top$ and $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$. Consider that the input is $u(t) = 2\sin(10t)$. Your plot should be a single plot versus time, since the norm function returns a scalar. Implement the whole setup on MATLAB via the `ode45` solver.

(c) How can you improve the convergence of the estimation error? Show that both analytically and via implementation.

(d) Most systems are noisy, in the sense that the real dynamics are more like:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$

where $w(t)$ is the noise. Often, we assume that $w(t)$ is a random variable with a well known distribution. Assume that $w(t)$ is a white Gaussian noise. You can implement this via the `randn` or `rand` commands in MATLAB. For example, you can write in the ODE solver file `w=0.1*randn(n,1)` which defines a vector of random quantities.

Investigate whether the observer you designed in the above parts is robust enough to this noise signal in the state evolution. Of course, you should not add noise in the observer dynamics, but only in the state dynamics. You can change the scaling factor 0.1 to something smaller or bigger—depending on the robustness of this observer.

Can you draw any conclusions?