Module 03
Linear Algebra Review & Solutions to State Space

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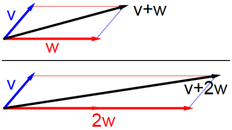
Vector Space and Matrix Properties

A (real) vector space V is a set with two operations:

- Vector sum $+ : V + V \to V$
- Scalar multiplication $\cdot : \mathbb{R} \times V \to V$

that has the following properties

1. Commutative: $x + y = y + x, \forall x, y \in V$
2. Associative: $(x + y) + z = x + (y + z), \forall x, y, z \in V$
3. Zero element: $\exists! 0 \in V$ such that $0 + x = x, \forall x \in V$
4. Inverse: $\forall x \in V, \exists (-x) \in V$ such that $x + (-x) = 0$
5. $(\alpha\beta)x = \alpha(\beta x), \forall \alpha, \beta \in \mathbb{R}, x \in V$
6. $\alpha(x + y) = \alpha x + \alpha y, \forall a \in \mathbb{R}, x, y \in V$
7. $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in \mathbb{R}, x \in V$



Examples of Vector Space

1. $\mathbb{R}^n$ with vector sum and scalar multiplication
2. $\mathbb{R}^{m \times n}$: the set of all m-by-n matrices
3. P_n: the set of all real polynomials in s with degree up to n:

$$\mathcal{P}_n := \{a_n s^n + \cdots + a_1 s + a_0 \mid a_0, \ldots, a_n \in \mathbb{R}\}$$

4. Give an index set $\mathcal{I}$, the set of all mappings from $\mathcal{I}$ to $\mathbb{R}^n$:

$$\mathcal{F}(\mathcal{I}; \mathbb{R}^n) := \{f : \mathcal{I} \to \mathbb{R}^n\}$$

5. $\{f : \mathbb{R}_+ \to \mathbb{R}^n \mid f \text{ is differentiable}\}$
6. The set of all functions $f(t)$, $t \geq 0$, with a Laplace transform
7. The set of all square integrable functions $f : \mathbb{R}_+ \to \mathbb{R}$
8. The set of all solutions $x(t) \in \mathbb{R}^n$, $t \geq 0$, to autonomous LTI system

$$\dot{x} = Ax, \quad x(0) = x_0$$

Matrix Rank

The rank of a matrix $A \in \mathbb{R}^{m\times n}$ is its maximum number of linearly independent columns (or rows), or equivalently, $\dim \mathcal{R}(A)$

- $\text{Rank}\,(A) \leq \min(m, n)$
- $\text{Rank}\,(A) = \text{Rank}\,(A^T)$
- $\text{Rank}\,(A) + \dim \mathcal{N}(A) = n$ (conservation of dimension)

Matrix $A \in \mathbb{R}^{m\times n}$ is full rank if $\text{Rank}\,(A) = \min(m, n)$, which means

- (for skinny matrices) independent column or injective maps
- (for fat matrices) independent rows or surjective maps
- (for square matrices) nonsingular or bijective maps

Inner Products

For $x, y \in \mathbb{R}^n$, their inner product is

$$\langle x, y \rangle := x^T y = x_1 y_1 + \cdots + x_n y_n$$

For $x, y, z \in \mathbb{R}^n$

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle x, y \rangle$
- $\langle x, x \rangle = \|x\|^2 \geq 0$, where $\|x\|$ is the Euclidean norm of x:

$$\|x\| := \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Theorem (Cauchy-Schwartz Inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{R}^n$$

Eigenvalues and Eigenvectors

Eigenvalues/Eigenvectors of a matrix

- Evales/vectors are **only defined for square**[1] **matrices**
- For a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, we always have n evalues/evectors
  - Some of these evalues might be distinct, real, repeated, imaginary
  - To find evalues($\boldsymbol{A}$), solve this equation ($\boldsymbol{I}_n$: identity matrix of size n)

$$\det(\lambda \boldsymbol{I}_n - A) = 0 \text{ or } \det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0 \Rightarrow a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

- **Example**: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$
- **Eigenvectors**: A number λ and a non-zero vector $\boldsymbol{v}$ satisfying

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \Rightarrow (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{v} = 0$$

  are called an eigenvalue and an eigenvector of $\boldsymbol{A}$
  - λ is an eigenvalue of an $n \times n$-matrix $\boldsymbol{A}$ if and only if $\lambda \boldsymbol{I}_n - \boldsymbol{A}$ is not invertible, which is equivalent to

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_\mathrm{n}) = 0.$$

[1] A square matrix has equal number of rows and columns.

Matrix Inverse

- Inverse of a generic 2by2 matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  – Notice that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I_2$

- Inverse of a generic 3by3 matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{T} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

$$\begin{array}{lll} A = (ei - fh) & D = -(bi - ch) & G = (bf - ce) \\ B = -(di - fg) & E = (ai - cg) & H = -(af - cd) \\ C = (dh - eg) & F = -(ah - bg) & I = (ae - bd) \end{array}$$

$$\boxed{\det(\mathbf{A}) = aA + bB + cC.}$$

  – Notice that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I_3$

Linear Algebra — Example 1

- Find the eigenvalues, eigenvectors, and inverse of matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

  - Eigenvalues: $\lambda_{1,2} = 5, -2$
  - Eigenvectors: $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \boldsymbol{v}_2 = \begin{bmatrix} -\frac{4}{3} & 1 \end{bmatrix}^\top$
  - Inverse: $\boldsymbol{A}^{-1} = -\frac{1}{10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$

- Write $\boldsymbol{A}$ in the matrix **diagonal transformation,** i.e., $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$ where $\boldsymbol{D}$ is the diagonal matrix containing the eigenvalues of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}^{-1}$$

  - **Only valid for matrices with distinct, real eigenvalues**

Rank of a Matrix

- Rank of a matrix: rank($\boldsymbol{A}$) is equal to the number of linearly independent rows or columns
  - **Example 1:** rank $\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}\right) = ?$
  - **Example 2:** rank $\left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}\right) = ?$
- Rank computation: reduce the matrix to a simpler form, generally row echelon form, by elementary row operations
  - **Example 2 Solution:**

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow 2r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow -3r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\rightarrow r_2 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow -2r_2 + r_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(\boldsymbol{A}) = 2$$

- **For a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:** rank($\boldsymbol{A}$) $\leq$ min(m, n)

Null Space of a Matrix

- The Null Space of any matrix $\boldsymbol{A}$ is the subspace $\mathcal{K}$ defined as follows:

$$\mathrm{N}(\boldsymbol{A}) = \mathrm{Null}(\boldsymbol{A}) = \ker(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathcal{K} | \boldsymbol{Ax} = \boldsymbol{0}\}$$

- $\mathrm{Null}(\boldsymbol{A})$ has the following three properties:
  - $\mathrm{Null}(\boldsymbol{A})$ always contains the zero vector, since $\boldsymbol{A0} = \boldsymbol{0}$
  - If $\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$ and $\boldsymbol{y} \in \mathrm{Null}(\boldsymbol{A})$, then $\boldsymbol{x} + \boldsymbol{y} \in \mathrm{Null}(\boldsymbol{A})$
  - If $\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$ and c is a scalar, then $c\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$
- **Example:** Find $\mathrm{N}(\boldsymbol{A})$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -4 & 2 & 3 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/16 & 0 \\ 0 & 1 & 13/8 & 0 \end{array}\right] \Rightarrow a = -\frac{1}{16}c, b = -\frac{13}{8}c \;\Rightarrow \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \begin{bmatrix} -1/16 \\ -13/8 \\ 1 \end{bmatrix} = \tilde{\alpha} \begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}}$$

Linear Algebra — Example 2

- Find the determinant, rank, and null-space set of this matrix:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

  - $\det(\boldsymbol{B}) = 0$
  - $\text{rank}(\boldsymbol{B}) = 2$
  - $\text{null}(\boldsymbol{B}) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall\, \alpha \in \mathbb{R}$
- Is there a relationship between the determinant and the rank of a matrix?
  - Yes! Matrix drops rank if determinant = zero ⇒ 1 zero evalue
- True or False?
  - $\boldsymbol{AB} = \boldsymbol{BA}$ for all $\boldsymbol{A}$ and $\boldsymbol{B}$—**FALSE!**
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible $\rightarrow (\boldsymbol{A} + \boldsymbol{B})$ is invertible—**FALSE!**

Matrix Exponential — 1

- Exponential of scalar variable:

$$e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots$$

- Power series converges $\forall\ a \in \mathbb{R}$
- How about matrices? For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, matrix exponential:

$$e^{\boldsymbol{A}} = \sum_{i=0}^{\infty} \frac{\boldsymbol{A}^i}{i!} = \boldsymbol{I}_n + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \frac{\boldsymbol{A}^4}{4!} + \cdots$$

- What if we have a time-variable t?

$$e^{t\boldsymbol{A}} = e^{\boldsymbol{A}t} = \sum_{i=0}^{\infty} \frac{(t\boldsymbol{A})^i}{i!} = \boldsymbol{I}_n + t\boldsymbol{A} + \frac{(t\boldsymbol{A})^2}{2!} + \frac{(t\boldsymbol{A})^3}{3!} + \frac{(t\boldsymbol{A})^4}{4!} + \cdots$$

Matrix Exponential Properties

For a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and a constant $t \in \mathbb{R}$:

1. $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \;\; \Rightarrow \;\; e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}$
2. [2] $\det(e^{\boldsymbol{A}t}) = e^{(\text{trace}(\boldsymbol{A}))t}$
3. $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$
4. $e^{\boldsymbol{A}^\top t} = (e^{\boldsymbol{A}t})^\top$
5. If $\boldsymbol{A}, \boldsymbol{B}$ commute, then: $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} = e^{\boldsymbol{B}t}e^{\boldsymbol{A}t}$
6. $e^{\boldsymbol{A}(t_1+t_2)} = e^{\boldsymbol{A}t_1}e^{\boldsymbol{A}t_2} = e^{\boldsymbol{A}t_2}e^{\boldsymbol{A}t_1}$

[2] Trace of a matrix is the sum of its diagonal entries.

When Is It Easy to Find $e^{\boldsymbol{A}}$? Method 1

Well...Obviously if we can directly use $e^{\boldsymbol{A}} = \boldsymbol{I}_n + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \cdots$

Three cases for Method 1

Case 1 $\boldsymbol{A}$ is nilpotent[3], i.e., $\boldsymbol{A}^k = 0$ for some k. Example:

$$\boldsymbol{A} = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

Case 2 $\boldsymbol{A}$ is idempotent, i.e., $\boldsymbol{A}^2 = \boldsymbol{A}$. Example:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Case 3 $\boldsymbol{A}$ is of rank one: $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$

$$\boldsymbol{A}^k = (\boldsymbol{v}^T\boldsymbol{u})^{k-1}\boldsymbol{A}, \ k = 1, 2, \ldots$$

[3] Any triangular matrix with 0s along the main diagonal is nilpotent

Method 2 — Diagonalizable Matrices

- Assume that $\boldsymbol{A}$ has distinct set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(\lambda_i \neq \lambda_j)$
- Then, as we learned from the diagonal transformation we can write

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}^{-1}$$

- You can now prove that

$$e^{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}^{-1}$$

Method 3 — Non-diagonalizable Matrices & Jordan Forms

- What if $\boldsymbol{A}$ does not have distinct set of eigenvalues?
- All matrices, whether diagonalizable or not, have a Jordan canonical form: $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1}$, then $e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{J}t}\boldsymbol{T}^{-1}$
- Generally, $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & & \\ & \ddots & \\ & & \boldsymbol{J}_p \end{bmatrix}$ $\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \Rightarrow$

$$e^{\boldsymbol{J}_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{n_i-1}e^{\lambda_i t}}{(n_i-1)!} \\ 0 & e^{\lambda_i t} & \ddots & \frac{t^{n_i-2}e^{\lambda_i t}}{(n_i-2)!} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \Rightarrow e^{\boldsymbol{A}t} = \boldsymbol{T} \begin{bmatrix} e^{\boldsymbol{J}_1 t} & & \\ & \ddots & \\ & & e^{\boldsymbol{J}_o t} \end{bmatrix} \boldsymbol{T}^{-1}$$

- We won't cover this topic in this course. Why?

Examples

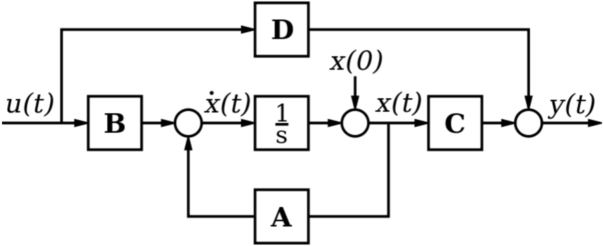
Find $e^{\boldsymbol{A}}$ and $e^{\boldsymbol{A}t}$ for the following matrices

Solution to the State-Space Equation

- In the next few slides, we'll answer this question: what is a solution to this vector-matrix first order ODE:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

- By solution, we mean a closed-form solution for $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ given:
  - An initial condition for the system, i.e., $\boldsymbol{x}(t_{initial}) = \boldsymbol{x}(0)$
  - A given control input signal, $\boldsymbol{u}(t)$, such as a step-input ($u(t) = 1$), ramp ($u(t) = t$), or anything else



The Curious Case of Autonomous Systems—Case 1

- Let's assume that we seek solution to this system first:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0 = \text{given} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned}$$

- This means that the system operates without any control input—**autonomous system** (e.g., autonomous vehicles)
- First, let's look at $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$—what's the solution to this first order ODE?
  - First case: $\boldsymbol{A} = a$ is a scalar $\Rightarrow x(t) = e^{at}x_0$
  - Second case: $\boldsymbol{A}$ is a matrix

$$\boxed{\Rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 \Rightarrow \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

- Exponential of scalars is very easy, but exponentials of matrices can be very challenging
- Hence, for an nth order system, where $n \geq 2$, we need to compute the matrix exponential in order to get a solution for the above system—we learned that in the linear algebra revision section

Example (Case 1)

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0, \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

- Find the solution for these two autonomous systems separately:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \boldsymbol{x}_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \boldsymbol{x}_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Note that this system is diagonalizable **(Case A)**
- If the system is not diagonalizable, we have to look for other methods to find the matrix exponential
- In particular, we have to find the Jordan form
- Anyway, let's find the state and output solutions now for this diagonalizable system
- **Solution:**

Case 2—Systems with Inputs

- MIMO (or SISO) LTI dynamical system:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \boldsymbol{x}(t_0) = \boldsymbol{x}_{t_0} = \text{given} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

- The to the above ODE is given by:

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\, d\tau}$$

- Clearly the output solution is:

$$\boxed{\boldsymbol{y}(t) = \underbrace{\boldsymbol{C}\left(e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0}\right)}_{\text{zero input response}} + \underbrace{\boldsymbol{C}\left(\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\, d\tau\right) + \boldsymbol{D}\boldsymbol{u}(t)}_{\text{zero state response}}}$$

- **Question:** how do I analytically compute $\boldsymbol{y}(t)$ and $\boldsymbol{x}(t)$?
- **Answer:** you need to (a) **integrate** and (b) **compute matrix exponentials** (given $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{x}_{t_0}, \boldsymbol{u}(t)$)

Example (Case 2)

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau$$

$$\boldsymbol{y}(t) = \underbrace{\boldsymbol{C}\left(e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0}\right)}_{\text{zero input response}} + \underbrace{\boldsymbol{C}\left(\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau\right) + \boldsymbol{D}\boldsymbol{u}(t)}_{\text{zero state response}}$$

- Find the solution for these two LTI systems with inputs:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \boldsymbol{x}_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, D_1 = 0, u_1(t) = 1$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \boldsymbol{x}_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, D_2 = 1, u_2(t) = 2e^{-2t}$$

- **Solution:**

Questions And Suggestions?

Thank You!

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