

# Module 04

## Linear Time-Varying Systems

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# Introduction to State Transition Matrix (STM)

- For the linear autonomous system

$$\dot{x}(t) = Ax(t), x(t_0) = x_0, t \geq 0$$

the state solution is

$$x(t) = e^{A(t-t_0)}x_0$$

- **Define the state transition matrix (STM):**

$$\phi(t, t_0) = e^{A(t-t_0)}$$

- STM ( $\phi(t, t_0)$ ) propagates an initial state along the LTI solution  $t$  time forward. Note that:

$$\phi(t_1 + t_2, t_0) = \phi(t_1, t_0)\phi(t_2, t_0) = \phi(t_2, t_0)\phi(t_1, t_0), \forall t_1, t_2 \geq 0$$

- In general, for an linear time varying system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), x(t_0) = x_0,$$

the state solution is given in terms of the STM:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

# Properties of the STM

For the linear autonomous system

$$\dot{x}(t) = Ax(t), x(t_0) = x_0, t \geq 0$$

the STM is:

$$\phi(t, t_0) = e^{A(t-t_0)}$$

- ①  $\phi(t_0, t_0) = \phi(t, t) = I$
- ②  $\phi^{-1}(t_1, t_2) = \phi(t_2, t_1)$
- ③  $\phi(t_1, t_2) = \phi(t_1, t_0)\phi(t_0, t_2)$
- ④  $\frac{d}{dt}(\phi(t, t_0)) = A\phi(t, t_0)$

Proofs:

# Solution Space and System Modes

- **Solution space**  $\mathcal{X}$  of the LTI system  $\dot{x}(t) = Ax(t)$  is the set of all its solutions:

$$\mathcal{X} := \{x(t), t \geq 0 \mid \dot{x} = Ax\}$$

- $\mathcal{X}$  is a vector space
- Dimension of  $\mathcal{X}$  is  $n$
- **System modes:** A **mode** of the LTI system  $\dot{x} = Ax$  is its solution from an eigenvector of  $A$ :

$$x(t) = e^{At} v_i = e^{\lambda_i t} v_i$$

- This is one property of the matrix exponential (see Module 3)
- The  $n$  (possibly repeated) **modes** form a basis of the solution space  $\mathcal{X}$

# Decomposition of State Solution

- Any state solution for an autonomous system can be written as a linear combination of **system modes**, assuming that  $A$  is diagonalizable
- This means that the solution space  $\mathcal{X}$  can be formed by these linear combinations
- $A = TDT^{-1}$  is assumed to be diagonalizable
- Assume that we start from  $x_0 = TT^{-1}x_0 = x_0$
- This means that we start from a linear combinations of  $v_i, i = 1, \dots, n$  since

$$x_0 = TT^{-1}x_0 = \sum_{i=1}^n (x_0^\top w_i) v_i = \sum_{i=1}^n \alpha_i v_i$$

where  $w_i$ 's are the rows of the  $T^{-1}$  matrix (or the left evecs)

- Given that construction, we can see that the solution  $x(t)$  is a LC of the modes  $e^{\lambda_i t} v_i$

# Changing Coordinates

- Changing of coordinates of an LTI system: basically means we're scaling the coordinates in a different way
- Assume that  $T \in \mathbb{R}^{n \times n}$  is a nonsingular transformation matrix
- Define  $\tilde{x} = T^{-1}x$ . Recall that  $\dot{x} = Ax + Bu$ , then:

$$\dot{\tilde{x}} = (T^{-1}AT)\tilde{x} + T^{-1}Bu = \tilde{A}\tilde{x} + \tilde{B}u$$

with initial conditions  $\tilde{x}(0) = T^{-1}x(0)$

- Remember the diagonal canonical form? We can get to it if the transformation  $T$  is the matrix containing the eigenvectors of  $A$
- What if the matrix is not diagonalizable? Well, we can still write  $A = TJT^{-1}$ , which means that  $\tilde{A} = J$  is the new state-space matrix via the eigenvector transformation
- In fact, you can show that if  $A = TJT^{-1}$  with  $j$  Jordan blocks (i.e.,  $J = \text{diag}(J_1, J_2, \dots, J_j)$ ), then after the transformation  $\tilde{x} = T^{-1}x$ , the LTI system becomes decoupled:

$$\dot{\tilde{x}}_1 = J_1\tilde{x}_1, \quad \dot{\tilde{x}}_2 = J_2\tilde{x}_2, \quad \dots, \quad \dot{\tilde{x}}_j = J_j\tilde{x}_j.$$

# STM of LTV Systems

- In the previous module, we learned how to compute the state and output solution
- We assumed that the system is time invariant, i.e.,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- What if the system is time varying:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \quad (*)$$

- How can we compute  $x(t)$  and  $y(t)$ ?
- That relies on finding the STM of the LTV system (\*)
- To do so, we have to find the exponential of a time-varying matrix

# STM of LTV Systems — 2

Theorem — STM of  $\dot{x}(t) = A(t)x(t)$

The STM of  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$  is given by

$$\phi(t, t_0) = \exp\left(\int_{t_0}^t A(q) dq\right)$$

if the following conditions are satisfied:

- 1  $A(t)$  has piecewise continuous entries for all  $t, t_0^a$
- 2  $A(t)$  commutes with its integral  $M(t, t_0) = \int_{t_0}^t A(q) dq$ , i.e.,  
 $A(t)M(t, t_0) = M(t, t_0)A(t)$

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<sup>a</sup>A function is piecewise continuous if: (a) it is defined throughout that interval, (b) its functions are continuous on that interval, and (c) there is no discontinuity at the endpoints of the defined interval.

- This theorem is very important, but can be very difficult to assess
- Consider a large system with TV  $A(t)$ . Then, numerical integration needs to be performed the check the conditions

# STM of LTV Systems — 3

- Given this analytical challenge, a natural question arises
- **What are easily testable** conditions that are sufficient for  $A(t)$  to commute with  $M(t, t_0)$ ?
- The following theorem investigates this question

## Theorem — STM Testing Conditions

$A(t)$  and  $M(t, t_0)$  commute if any of the following conditions hold:

- 1  $A(t) = A$  is a constant matrix
- 2  $A(t) = \beta(t)A$  where  $\beta(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is a scalar function and  $A$  is a constant matrix
- 3  $A(t) = \sum_{i=1}^m \beta_i(t)A_i$  where  $\beta_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  are all scalar functions and  $A_i$ 's are all constant matrices that commute with each other, i.e.,  $A_i A_j = A_j A_i, \forall i, j \in \{1, 2, \dots, m\}$
- 4 There exists a factorization  $A(t) = TD(t)T^{-1}$  where  $D(t) = \text{diag}(\lambda_1(t), \dots, \lambda_n(t))$

# Example 1

- $A(t) = \dot{\alpha}(t) \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$
- What is the state transition matrix?
- **Solution:** notice that  $A(t)$  fits with the second characterization, hence  $A(t)$  and  $M(t, t_0)$  commute (assume that  $\alpha(t)$  is continuous differentiable function)
- Note that  $A$  is nilpotent of order 2
- Solution:

$$\phi(t, t_0) = \exp \left( \int_{t_0}^t A(q) dq \right) = I + \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} (\alpha(t) - \alpha(t_0))$$

## Example 2

- $A(t) = \begin{bmatrix} \dot{a}(t) & \dot{b}(t) \\ \dot{b}(t) & \dot{a}(t) \end{bmatrix}$ , find the STM

- Note that  $A(t) = \dot{a}(t)I + \dot{b}(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- We can apply the third case to obtain:

$$\phi(t, t_0) = \exp\left(\int_{t_0}^t A(q) dq\right) = e^{a(t)-a(t_0)} \exp\left(\begin{bmatrix} 0 & b(t) - b(t_0) \\ b(t) - b(t_0) & 0 \end{bmatrix}\right)$$

- Recall that if

$$A_2 = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \Rightarrow e^{A_2 t} = \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}$$

- Hence,

$$\phi(t, t_0) = e^{a(t)-a(t_0)} \begin{bmatrix} \cosh(b(t) - b(t_0)) & \sinh(b(t) - b(t_0)) \\ \sinh(b(t) - b(t_0)) & \cosh(b(t) - b(t_0)) \end{bmatrix}$$

# Overall Solution

- So, given that we have the state transition matrix, how can we find the overall solution of the LTV system?
- The answer is simple:

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau)d\tau$$

# More Examples on STM Computations

Find the state transition matrix of

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

# Questions And Suggestions?



**Thank You!**

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