Module 08
Observability and State Estimator Design of Dynamical LTI Systems

Ahmad F. Taha

EE 5143: Linear Systems and Control

Email: ahmad.taha@utsa.edu

Webpage: http://engineering.utsa.edu/ataha

November 21, 2017

Introduction to Observability

- **Observability:** The ability to observer what's happening inside your system (i.e., to know system states $x(t)$)
- Observability: In order to see what is going on inside the system under observation (i.e., output $y(t)$), **the system must be observable**. Observation: output $y(t)$
- Given this dynamical system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, \\ y(k) &= Cx(k) + Du(k), \\ \text{or } \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  a natural question arises: can we learn anything about $x(t)$ given $y(t)$ and $u(t)$ for a specific time t?
- Clearly, if we know $x(0)$ and $u(t)$ for all t, we can determine $x(t)$ via

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

- However, if $x(0)$ if unknown, can you obtain $x(t)$ via only $y(t), u(t)$?

Observability — 1

DTLTI system (n states, m inputs, p outputs):

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, & (1) \\ y(k) &= Cx(k) + Du(k), & (2) \end{aligned}$$

- **Application**: **given that** A, B, C, D**, and** $u(k), y(k)$ **are known** $\forall k = 0 : 1 : k-1$, **can we determine** $x(0)$?
- **Solution:**

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Observability — 2

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$Y(k-1) = \mathcal{O}_k x(0) + \mathcal{T}_k U(k-1) \implies \mathcal{O}_k x(0) = Y(k-1) - \mathcal{T}_k U(k-1)$$

- Since $\mathcal{O}_k, \mathcal{T}_k, Y(k-1), U(k-1)$ are all known quantities, then we can find a unique $x(0)$ iff $\mathcal{O}_k$ is full rank

Observability Definition

DTLTI system **is observable at time** k if the initial state $x(0)$ can be uniquely determined from any given

$$u(0), \dots, u(k-1), y(0), \dots, y(k-1).$$

Quantifying Observability

Observability Test

For a system with n states and p outputs, the test for observability is that matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$ has full column rank (i.e., rank$(\mathcal{C}) = n$).

The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

1. $\mathcal{O}$ is full rank, system is observable
2. PBH Test: for any $\lambda \in \mathbb{C}$, rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$
3. Eigenvector Test: for any right evector of A, $Cv_i \neq 0$
4. The following matrices are nonsingular

$$\sum_{i=0}^{n-1} (A^\top)^i C^\top C A^i \quad \text{(DTLIT)} \quad \& \quad \int_0^t e^{A^\top \tau} C^\top C e^{A\tau}\, d\tau \quad \text{(CTLTI)}$$

Example 1

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is this system controllable?
- Is this system observable?
- **Answers:** Yes, Yes!
- MATLAB commands: `ctrb, obsv`

Example 2

Determine whether the following system is observable or not:

$$x(k+1) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 \end{bmatrix} x(k).$$

The challenge here is to be able to figure out which test should be used. Clearly, A has 7 evalues as follows: $\lambda_A = \{-1, -1, -1, -1, 0, 0, 0\}$. Test 2 is the easiest test to use here. Applying the test, you'll see that the PBH test fails for the zero eigenvalue, which means that the system is not observable.

Unobservable Subspace

- **Unobservable subspace: null-space of $\mathcal{O}_k = \mathcal{N}(\mathcal{O}_k)$**
- It is basically the space (i.e., set of states $x \in \mathcal{X}$ that you cannot estimate or observer
- Notice that if $x(0) \in Null(\mathcal{O}_k)$, and $u(k) = 0$, then the output is going to zero from $[0, k-1]$
- Notice that input $u(k)$ does not impact the ability to determine $x(0)$
- The unobservable subspace $\mathcal{N}(\mathcal{O}_k)$ is A-invariant: if $z \in \mathcal{N}(\mathcal{O}_k)$, then $Az \in \mathcal{N}(\mathcal{O}_k)$

Unobservable Space

The null spaces $Null(\mathcal{O}_k) = \mathcal{N}(\mathcal{O}_k)$ satisfy

$$\mathcal{N}(\mathcal{O}_0) \supseteq \mathcal{N}(\mathcal{O}_1) \supseteq \cdots \supseteq \mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_{n+1}) = \cdots$$

This means that the more output measurements you have, the smaller the unobservable subspace.
It also implies that you cannot get more information if you go above $k > n$. You can prove this by C-H theorem ($A^n = \sum_{i=0}^{n-1} \alpha_i A^i$)

Detectability

Detectability Definition

DTLTI or CTLIT system, defined by (A, C), is detectable if there exists a matrix L such that $A - LC$ is stable.

Detectability Theorem

DTLTI or CTLIT system, defined by (A, C) is detectable if all its unobservable modes correspond to stable eigenvalues of A.

Facts:

- A is stable $\Rightarrow (A, C)$ is detectable
- (A, C) is observable $\Rightarrow (A, C)$ is detectable as well
- (A, B) is not observable $\Rightarrow$ it could still be detectable
- If system has some unobservable modes that are unstable, then no gain L can make $A - LC$ stable
- $\Rightarrow$ Observer will fail to track system state

Observability for CT Systems

- The previous derivation for observability was for DT LTI systems
- What if we have a CT LTI system? Do we obtain the same observability testing conditions?
- Yes, we do!
- First, note that the control input $u(t)$ plays no role in observability, just like how the output $y(t)$ plays no role in controllability
- To see that, consider the following system with n states, p outputs, where (again) we want to obtain $x(t_0)$ (unknown):

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t) \quad x(t_0) = x_0 \Longrightarrow$$

$$\begin{aligned} y(t_0) &= Cx(t_0) \\ \dot{y}(t_0) &= C\dot{x}(t_0) = CAx(t_0) \\ \ddot{y}(t_0) &= C\ddot{x}(t_0) = CA^2x(t_0) \\ &\vdots \\ y^{(n-1)}(t_0) &= Cx^{(n-1)}(t_0) = CA^{n-1}x(t_0) \end{aligned}$$

Observability for CT LTI Systems — 2

- We can write the previous equation as:

$$\begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \ddot{y}(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{bmatrix} = Y(t_0) = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{=\mathcal{O}\in\mathbb{R}^{np\times n}} x(t_0) \Rightarrow$$

$$x(t_0) = \mathcal{O}^{\dagger} Y(t_0) = (\mathcal{O}^{\top}\mathcal{O})^{-1}\mathcal{O}^{\top} Y(t_0)$$

- Hence, the initial conditions can be determined if the observability matrix is full column rank
- This condition is identical to the DT case where we also wanted to obtain $x(k=0)$ from a set of output measurements
- The difference here is that we had to obtain derivatives of the output at t_0
- Can you rederive the equations if $u(t) \neq 0$? It won't make an impact on whether a solution exists, but it'll change $x(t_0)$

Controllability-Observability Duality, Minimality

Duality

The CT LTI system with state-space matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is called the **dual** of another CT LTI system with state-space matrices (A, B, C, D) if

$$\tilde{A} = A^\top, \;\; \tilde{B} = C^\top, \;\; \tilde{C} = B^\top, \;\; \tilde{D} = D^\top.$$

Controllability-Observability Duality

CT system (A, B, C, D) is observable (controllable) if and only if its dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is controllable (observable).

Minimality

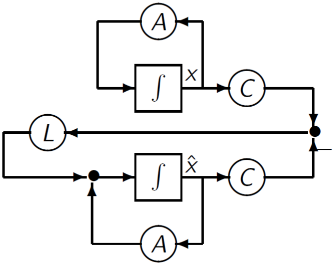
A system (A, B, C, D) is called minimal if and only if it is both controllable and observable.

Duality and minimality also holds for DT-LTI systems.

Observer Design

Original system with unknown $x(0)$:

$$\dot{x} = Ax,$$
$$y = Cx$$



Simulator with linear feedback:

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \quad \hat{x}(0) = 0$$
$$\hat{y} = C\hat{x}$$

- Objective here is to estimate (in real-time) the state of the actual system $x(t)$ given that ICs $x(0)$ are not known
- To do that, we design an observer—dynamic state estimator (DSE)
- Define dynamic estimation error: $e(t) = x(t) - \hat{x}(t)$
- Error dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t)$$

- Hence, $e(t) \to 0$, as $t \to \infty$ if $\texttt{eig}(A - LC) < 0$
- **Objective:** design observer/estimator gain L such that $\texttt{eig}(A - LC) < 0$ or at a certain location

Example — Controller Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Is the system stable? What are the eigenvalues?
- **Solution:** unstable, $\texttt{eig}(A) = 4, -2$
- Find linear state-feedback gain K (i.e., $u = -Kx$), such that the poles of the closed-loop controlled system are -3 and -5
- Characteristic polynomial: $\lambda^2 + (k_1 - 2)\lambda + (3k_2 - k_1 - 8) = 0$
- **Solution:** $u = -Kx = -[10 \;\; 11] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -10x_1 - 11x_2$
- MATLAB command: $K = \texttt{place(A,B,eig\_desired)}$
- What if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, can we stabilize the system?

Example — Observer Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$
- Find linear state-observer gain $L = [l_1 \;\; l_2]^\top$ such that the poles of the estimation error are -5 and -3
- Characteristic polynomial:
  $\lambda^2 + (-2 + l_2 + 0.5l_1)\lambda + (-8 + 0.5l_2 + 2.5l_1) = 0$
- **Solution:** $L = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
- MATLAB command: $L =$ `place(A',C',eig_desired)`

Observer, Controller Design for DT Systems—Summary

- For CT system
$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$
  - To design a stabilizing controller, find K such that
$$eig(A_{cl}) = eig(A - BK) < 0$$
    or at a prescribed location
  - To design a converging estimator (observer), find L such that
$$eig(A_{cl}) = eig(A - LC) < 0$$
    or at a prescribed location
- What if the system is DT?
$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k)$$
  - To design a stabilizing controller, find K such that
$$-1 < eig(A_{cl}) = eig(A - BK) < 1 \text{ or at a prescribed location}$$
  - To design a converging estimator (observer), find L such that
$$-1 < eig(A_{cl}) = eig(A - LC) < 1 \text{ or at a prescribed location}$$

Observer Design

- What if the system dynamics are:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

- The observer dynamics will then be:

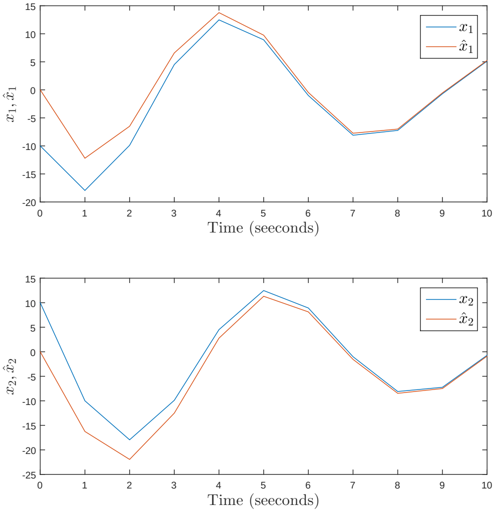
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$$

- Hence, the control input shouldn't impact the estimation error
- Why? Because the input $u(t)$ is know!
- Estimation error:

$$e(t) = x(t) - \hat{x}(t) \Longrightarrow \dot{e}(t) = \dot{x}(t) = \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t))$$

$$\Longrightarrow \dot{e}(t) = (A - LC)e(t)$$

MATLAB Example

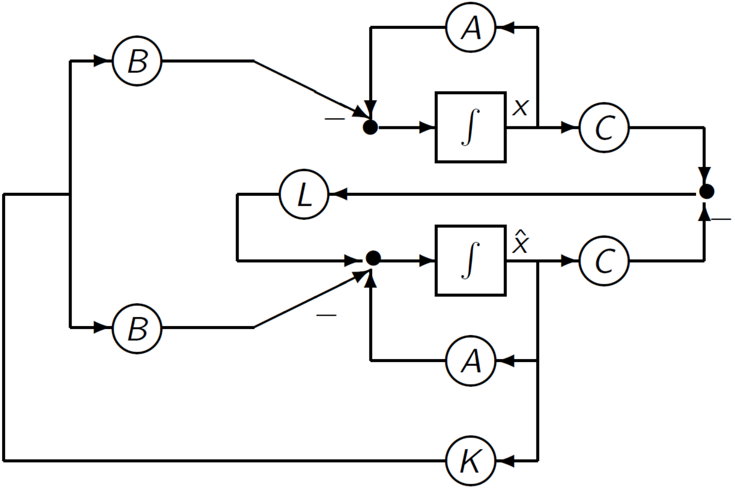



```
A=[1 -0.8; 1 0];
B=[0.5; 0];
C=[1 -1];
% Selecting desired poles
eig_desired=[.5 .7];
L=place(A',C',eig_desired)';
% Initial state
x=[-10;10];
% Initial estimate
xhat=[0;0];
% Dynamic Simulation
XX=x;
XXhat=xhat;
T=10;
% Constant Input Signal
UU=.1*ones(1,T);
for k=0:T-1,
u=UU(k+1);
y=C*x;
yhat=C*xhat;
x=A*x+B*u;
xhat=A*xhat+B*u+L*(y-yhat);
XX=[XX,x];
XXhat=[XXhat,xhat];
end
% Plotting Results
subplot(2,1,1)
plot(0:T,[XX(1,:);XXhat(1,:)]);
subplot(2,1,2)
plot(0:T,[XX(2,:);XXhat(2,:)]);
```

Observer-Based Control — 1

- Recall that for LSF control: $u(t) = -Kx(t)$
- What if $x(t)$ is not available, i.e., it can only be estimated?
- **Solution:** get $\hat{x}$ by designing L
- Apply LSF control using $\hat{x}$ with a LSF matrix K to both the original system and estimator
- **Question:** how to design K and L simultaneously? Poles of the closed-loop system?
- This is called an observer-based controller (OBC)
- Design questions: how shall we design K and L? Are these designs independent?

Observer-Based Control — 2



Notice that $u(t) = -K\hat{x}(t)$

Observer-Based Control — 3

- Closed-loop dynamics:

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BK\hat{x}(t)\\ \dot{\hat{x}}(t) &= A\hat{x}(t) + L(y(t) - \hat{y}(t)) - BK\hat{x}(t)\end{aligned}$$

- The overall system (observer + controller) can be written as follows:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & -BK\\ LC & A-LC-BK\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

- Transformation: $\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} = \begin{bmatrix}x(t)\\ x(t)-\hat{x}(t)\end{bmatrix} = \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$
- Hence, we can write:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A-BK & BK\\ 0 & A-LC\end{bmatrix}}_{A_{\text{cl}}}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}$$

- If the system is controllable & observable $\Rightarrow$ $\texttt{eig}(A_{\text{cl}})$ can be arbitrarily assigned by proper K and LWhat if the system is stabilizable and detectable?

Separation Principle

$$\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A-BK & BK\\ 0 & A-LC\end{bmatrix}}_{A_{\text{cl}}}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} \quad \equiv \quad \begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & -BK\\ LC & A-LC-BK\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

- Notice the above dynamics for the OBC are equivalent
- What are the evalues of the closed loop system above?
- Since A_{cl} is block diagonal, the evalues of A_{cl} are

$$eig(A-BK)\bigcup eig(A-LC)$$

- $eig(A-BK)$ characterizes the **state control dynamics**
- $eig(A-BK)$ characterizes the **state estimation dynamics**
- If the system is obsv. **AND** cont. $\Longrightarrow$ evalues(A_{cl}) can be arbitrarily assigned by properly designing K and L
- If the system is detect. **AND** stab. $\Longrightarrow$ evalues(A_{cl}) can be stabilized via properly designing K and L

OBC Example

Design an OBC (i.e., $u(t) = -K\hat{x}(t)$) for the following SISO system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

1. Before doing anything, check whether system is cont. (or stab.) and obs. (or det.): **system is cont. AND obs.**
2. First, design a stabilizing state feedback control, i.e., find K s.t.

   $$eig(A-BK) < 0, A-BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \Rightarrow K = \begin{bmatrix} 4 & 2 \end{bmatrix} \text{ does the job}$$

3. Second, design a stabilizing observer (estimator), i.e., find L s.t.

   $$eig(A-LC) < 0, A-LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 10 & 100 \end{bmatrix}^\top \text{ does the job}$$

4. Finally, overall system design:

$$\begin{aligned} u(t) &= -K\hat{x}(t) = -4\hat{x}_1(t) - 2\hat{x}_2(t) \\ \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + 10(y(t) - \hat{x}_1(t)) \\ \dot{\hat{x}}_2(t) &= u(t) + 100(y(t) - \hat{x}_1(t)) \end{aligned}$$

Questions And Suggestions?

Thank You!

Please visit

engineering.utsa.edu/ataha

IFF you want to know more ☺