

Module 09

Observability and State Estimator Design of Dynamical LTI Systems

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Introduction to Observability

- **Observability:** The ability to observe what's happening inside your system (i.e., to know system states $x(t)$)
- Observability: In order to see what is going on inside the system under observation (i.e., output $y(t)$), **the system must be observable**. Observation: output $y(t)$
- Given this dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0,$$

$$y(k) = Cx(k) + Du(k),$$

$$\text{or } \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

$$y(t) = Cx(t) + Du(t)$$

a natural question arises: can we learn anything about $x(t)$ given $y(t)$ and $u(t)$ for a specific time t ?

- Clearly, if we know $x(0)$ and $u(t)$ for all t , we can determine $x(t)$ via

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- However, if $x(0)$ is unknown, can you obtain $x(t)$ via only $y(t)$, $u(t)$?

Observability — 1

DTLTI system (n states, m inputs, p outputs):

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad (1)$$

$$y(k) = Cx(k) + Du(k), \quad (2)$$

- **Application:** given that A, B, C, D , and $u(k), y(k)$ are known $\forall k = 0 : 1 : k - 1$, **can we determine** $x(0)$?
- **Solution:**

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Observability — 2

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$Y(k-1) = \mathcal{O}_k x(0) + \mathcal{T}_k U(k-1) \implies \mathcal{O}_k x(0) = Y(k-1) - \mathcal{T}_k U(k-1)$$

- Since $\mathcal{O}_k, \mathcal{T}_k, Y(k-1), U(k-1)$ are all known quantities, then we can find a unique $x(0)$ iff \mathcal{O}_k is full rank

Observability Definition

DTLTI system is **observable at time k** if the initial state $x(0)$ can be uniquely determined from any given

$$u(0), \dots, u(k-1), y(0), \dots, y(k-1).$$

Quantifying Observability

Observability Test

For a system with n states and p outputs, the test for observability is that

matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$ has full column rank (i.e., $\text{rank}(\mathcal{O}) = n$).

The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

- ① \mathcal{O} is full rank, system is observable
- ② PBH Test: for any $\lambda \in \mathbb{C}$, $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$
- ③ Eigenvector Test: for any right evector of A , $Cv_i \neq 0$
- ④ The following matrices are nonsingular

$$\sum_{i=0}^{n-1} (A^T)^i C^T C A^i \quad (\text{DTLTI}) \quad \& \quad \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau \quad (\text{CTLTI})$$

Example 1

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is this system controllable?
- Is this system observable?
- **Answers:** Yes, Yes!
- MATLAB commands: `ctrb`, `obsv`

Example 2

Determine whether the following system is observable or not:

$$x(k+1) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 \end{bmatrix} x(k).$$

The challenge here is to be able to figure out which test should be used. Clearly, A has 7 evalues as follows: $\lambda_A = \{-1, -1, -1, -1, 0, 0, 0\}$. Test 2 is the easiest test to use here. Applying the test, you'll see that the PBH test fails for the zero eigenvalue, which means that the system is not observable.

Unobservable Subspace

- **Unobservable subspace: null-space of $\mathcal{O}_k = \mathcal{N}(\mathcal{O}_k)$**
- It is basically the space (i.e., set of states $x \in \mathcal{X}$ that you cannot estimate or observer
- Notice that if $x(0) \in \text{Null}(\mathcal{O}_k)$, and $u(k) = 0$, then the output is going to zero from $[0, k - 1]$
- Notice that input $u(k)$ does not impact the ability to determine $x(0)$
- The unobservable subspace $\mathcal{N}(\mathcal{O}_k)$ is A -invariant: if $z \in \mathcal{N}(\mathcal{O}_k)$, then $Az \in \mathcal{N}(\mathcal{O}_k)$

Unobservable Space

The null spaces $\text{Null}(\mathcal{O}_k) = \mathcal{N}(\mathcal{O}_k)$ satisfy

$$\mathcal{N}(\mathcal{O}_0) \supseteq \mathcal{N}(\mathcal{O}_1) \supseteq \dots \supseteq \mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_{n+1}) = \dots$$

This means that the more output measurements you have, the smaller the unobservable subspace.

It also implies that you cannot get more information if you go above $k > n$. You can prove this by C-H theorem ($A^n = \sum_{i=0}^{n-1} \alpha_i A^i$)

Detectability

Detectability Definition

DTLTI or CTLIT system, defined by (A, C) , is detectable if there exists a matrix L such that $A - LC$ is stable.

Detectability Theorem

DTLTI or CTLIT system, defined by (A, C) is detectable if all its unobservable modes correspond to stable eigenvalues of A .

Facts:

- A is stable $\Rightarrow (A, C)$ is detectable
- (A, C) is observable $\Rightarrow (A, C)$ is detectable as well
- (A, B) is not observable \Rightarrow it could still be detectable
- If system has some unobservable modes that are unstable, then no gain L can make $A - LC$ stable
- \Rightarrow Observer will fail to track system state

Observability for CT Systems

- The previous derivation for observability was for DT LTI systems
- What if we have a CT LTI system? Do we obtain the same observability testing conditions?
- Yes, we do!
- First, note that the control input $u(t)$ plays no role in observability, just like how the output $y(t)$ plays no role in controllability
- To see that, consider the following system with n states, p outputs, where (again) we want to obtain $x(t_0)$ (unknown):

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t) \quad x(t_0) = x_0 \implies$$

$$y(t_0) = Cx(t_0)$$

$$\dot{y}(t_0) = C\dot{x}(t_0) = CAx(t_0)$$

$$\ddot{y}(t_0) = C\ddot{x}(t_0) = CA^2x(t_0)$$

$$\vdots$$

$$y^{(n-1)}(t_0) = Cx^{(n-1)}(t_0) = CA^{n-1}x(t_0)$$

Observability for CT LTI Systems — 2

- We can write the previous equation as:

$$\begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \ddot{y}(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{bmatrix} = Y(t_0) = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{= \mathcal{O} \in \mathbb{R}^{np \times n}} x(t_0) \Rightarrow$$

$$x(t_0) = \mathcal{O}^\dagger Y(t_0) = (\mathcal{O}^\top \mathcal{O})^{-1} \mathcal{O}^\top Y(t_0)$$

- Hence, the initial conditions can be determined if the observability matrix is full column rank
- This condition is identical to the DT case where we also wanted to obtain $x(k=0)$ from a set of output measurements
- The difference here is that we had to obtain derivatives of the output at t_0
- Can you rederive the equations if $u(t) \neq 0$? It won't make an impact on whether a solution exists, but it'll change $x(t_0)$

Controllability-Observability Duality, Minimality

Duality

The CT LTI system with state-space matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is called the **dual** of another CT LTI system with state-space matrices (A, B, C, D) if

$$\tilde{A} = A^T, \quad \tilde{B} = C^T, \quad \tilde{C} = B^T, \quad \tilde{D} = D^T.$$

Controllability-Observability Duality

CT system (A, B, C, D) is observable (controllable) if and only if its dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is controllable (observable).

Minimality

A system (A, B, C, D) is called minimal if and only if it is both controllable and observable.

Duality and minimality also holds for DT-LTI systems.

Observer Design

Original system with unknown $x(0)$:

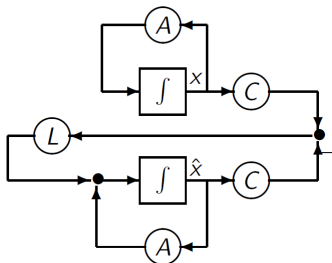
$$\dot{x} = Ax,$$

$$y = Cx$$

Simulator with linear feedback:

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \quad \hat{x}(0) = 0$$

$$\hat{y} = C\hat{x}$$



- Objective here is to estimate (in real-time) the state of the actual system $x(t)$ given that ICs $x(0)$ are not known
- To do that, we design an observer—dynamic state estimator (DSE)
- Define dynamic estimation error: $e(t) = x(t) - \hat{x}(t)$
- Error dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t)$$

- Hence, $e(t) \rightarrow 0$, as $t \rightarrow \infty$ if $\text{eig}(A - LC) < 0$
- **Objective:** design observer/estimator gain L such that $\text{eig}(A - LC) < 0$ or at a certain location

Example — Controller Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Is the system stable? What are the eigenvalues?
- **Solution:** unstable, $\text{eig}(A) = 4, -2$
- Find linear state-feedback gain K (i.e., $u = -Kx$), such that the poles of the closed-loop controlled system are -3 and -5
- Characteristic polynomial: $\lambda^2 + (k_1 - 2)\lambda + (3k_2 - k_1 - 8) = 0$
- **Solution:** $u = -Kx = -[10 \ 11] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -10x_1 - 11x_2$
- MATLAB command: $K = \text{place}(A, B, \text{eig_desired})$
- What if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, can we stabilize the system?

Example — Observer Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $C = [0.5 \quad 1]$
- Find linear state-observer gain $L = [l_1 \quad l_2]^T$ such that the poles of the estimation error are -5 and -3
- Characteristic polynomial:
$$\lambda^2 + (-2 + l_2 + 0.5l_1)\lambda + (-8 + 0.5l_2 + 2.5l_1) = 0$$
- **Solution:** $L = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
- MATLAB command: $L = \text{place}(A', C', \text{eig_desired})$

Observer, Controller Design for DT Systems—Summary

- For CT system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

- To design a stabilizing controller, find K such that

$$\text{eig}(A_{cl}) = \text{eig}(A - BK) < 0$$

or at a prescribed location

- To design a converging estimator (observer), find L such that

$$\text{eig}(A_{cl}) = \text{eig}(A - LC) < 0$$

or at a prescribed location

- What if the system is DT?

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k)$$

- To design a stabilizing controller, find K such that

$$-1 < \text{eig}(A_{cl}) = \text{eig}(A - BK) < 1 \quad \text{or at a prescribed location}$$

- To design a converging estimator (observer), find L such that

$$-1 < \text{eig}(A_{cl}) = \text{eig}(A - LC) < 1 \quad \text{or at a prescribed location}$$

Observer Design

- What if the system dynamics are:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

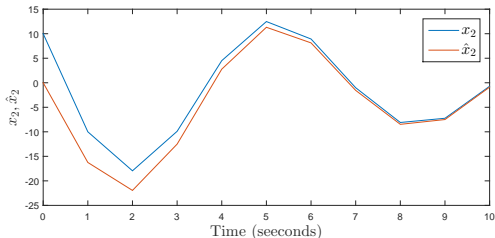
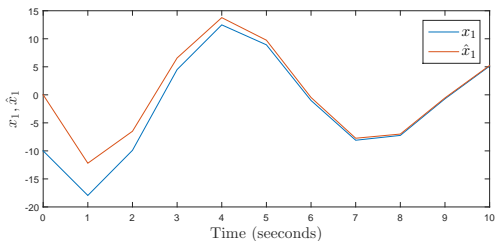
- The observer dynamics will then be:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$$

- Hence, the control input shouldn't impact the estimation error
- Why? Because the input $u(t)$ is known!
- Estimation error:

$$\begin{aligned} e(t) = x(t) - \hat{x}(t) &\implies \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) \\ &\implies \dot{e}(t) = (A - LC)e(t) \end{aligned}$$

MATLAB Example



```

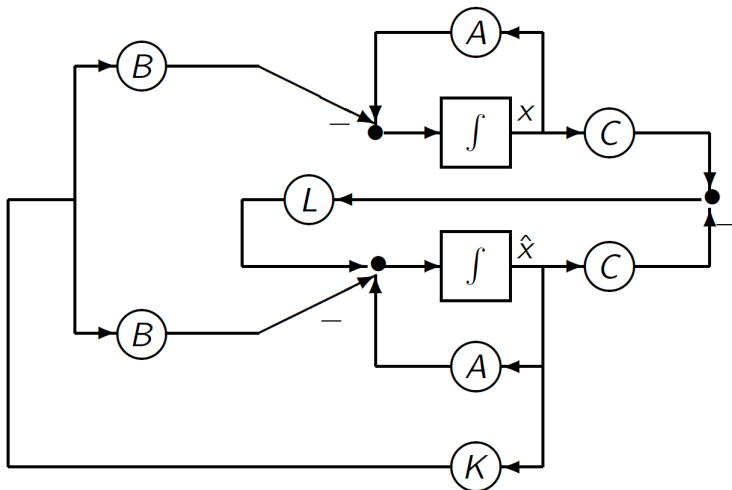
A=[1 -0.8; 1 0];
B=[0.5; 0];
C=[1 -1];
% Selecting desired poles
eig_desired=[.5 .7];
L=place(A',C',eig_desired)';
% Initial state
x=[-10;10];
% Initial estimate
xhat=[0;0];
% Dynamic Simulation
XX=x;
XXhat=xhat;
T=10;
% Constant Input Signal
UU=-.1*ones(1,T);
for k=0:T-1,
    u=UU(k+1);
    y=C*x;
    yhat=C*xhat;
    x=A*x+B*u;
    xhat=A*xhat+B*u+L*(y-yhat);
    XX=[XX,x];
    XXhat=[XXhat,xhat];
end
% Plotting Results
subplot(2,1,1)
plot(0:T,[XX(1,:);XXhat(1,:)]);
subplot(2,1,2)
plot(0:T,[XX(2,:);XXhat(2,:)]);

```

Observer-Based Control — 1

- Recall that for LSF control: $u(t) = -Kx(t)$
- What if $x(t)$ is not available, i.e., it can only be estimated?
- **Solution:** get \hat{x} by designing L
- Apply LSF control using \hat{x} with a LSF matrix K to both the original system and estimator
- **Question:** how to design K and L simultaneously? Poles of the closed-loop system?
- This is called an observer-based controller (OBC)
- Design questions: how shall we design K and L ? Are these designs independent?

Observer-Based Control — 2



Notice that $u(t) = -K\hat{x}(t)$

Observer-Based Control — 3

- Closed-loop dynamics:

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - \hat{y}(t)) - BK\hat{x}(t)$$

- The overall system (observer + controller) can be written as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

- Transformation: $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$
- Hence, we can write:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

- If the system is controllable & observable $\Rightarrow \text{eig}(A_{cl})$ can be arbitrarily assigned by proper K and L What if the system is stabilizable and detectable?

Separation Principle

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \equiv \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

- Notice the above dynamics for the OBC are equivalent
- What are the values of the closed loop system above?
- Since A_{cl} is block diagonal, the values of A_{cl} are

$$eig(A - BK) \cup eig(A - LC)$$

- $eig(A - BK)$ characterizes the **state control dynamics**
- $eig(A - LC)$ characterizes the **state estimation dynamics**
- If the system is obsv. **AND** cont. \implies values(A_{cl}) can be arbitrarily assigned by properly designing K and L
- If the system is detect. **AND** stab. \implies values(A_{cl}) can be stabilized via properly designing K and L

OBC Example

Design an OBC (i.e., $u(t) = -K\hat{x}(t)$) for the following SISO system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = [1 \quad 0] x(t)$$

- 1 Before doing anything, check whether system is cont. (or stab.) and obs. (or det.): **system is cont. AND obs.**
- 2 First, design a stabilizing state feedback control, i.e., find K s.t.

$$\text{eig}(A-BK) < 0, A-BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \Rightarrow K = [4 \quad 2] \text{ does the job}$$

- 3 Second, design a stabilizing observer (estimator), i.e., find L s.t.

$$\text{eig}(A-LC) < 0, A-LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \Rightarrow L = [10 \quad 100]^T \text{ does the job}$$

- 4 Finally, overall system design:

$$\begin{aligned} u(t) &= -K\hat{x}(t) = -4\hat{x}_1(t) - 2\hat{x}_2(t) \\ \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + 10(y(t) - \hat{x}_1(t)) \\ \dot{\hat{x}}_2(t) &= u(t) + 100(y(t) - \hat{x}_1(t)) \end{aligned}$$

Questions And Suggestions?



Thank You!

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IFF you want to know more 😊