If $E = I$, then (4.2a) becomes, with $N = n - 1$,

$$\left[\begin{array}{cc} 0 & F^{n-1} \\ 0 & I \\ \hline 0 & HF^{n-2} \\ \vdots & \vdots \\ 0 & HF \\ 0 & H \end{array}\right] \begin{bmatrix} x_{n-1} \\ w_0 \end{bmatrix} = \left[\begin{array}{c} x_n \\ Fx_0 \\ \hline \bar{y}_{1,n} \end{array}\right]$$

so that $w_0 = Fx_0$. Taking into account this and (4.2b), write the above as

$$\left[\begin{array}{c} F^n \\ I \\ \hline V_n^s \end{array}\right] x_0 \triangleq \left[\begin{array}{c} F^n \\ I \\ \hline HF^{n-1} \\ \vdots \\ HF \\ H \end{array}\right] x_0 = \left[\begin{array}{c} x_n \\ x_0 \\ \hline \bar{y}_{0,n} \end{array}\right], \tag{4.5}$$

i.e., the familiar state-space result. Note that the usual definitions of observability and reconstructibility for discrete state systems both derive from our single definition for observability of the pair (x_0, x_n). Hence, in the usual terminology, x_0 is observable if (4.5) has a unique solution with respect to x_0, i.e., $\mathfrak{N}(V_n^s) = 0$. On the other hand, x_n is reconstructible if (4.5) has a unique solution with respect to x_n, i.e., $\mathfrak{N}(V_n^s) \subset \mathfrak{N}(F^n)$.

Finally, the descriptor variable boundary value reconstruction problem is solved.

Corollary 4.4: Let (2.1) be regular and $\bar{u}_{0,N+1} = 0$. Then the values of Fx_0 and Ex_{N+1} can be uniquely determined from $\bar{y}_{1,N+1}$ for all admissible (x_0, x_{N+1}) if and only if (4.3) holds. In this case, they are given by

$$\begin{bmatrix} Ex_{N+1} \\ Fx_0 \end{bmatrix} = \begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{bmatrix} V_N^+ \bar{y}_{1,N+1}. \tag{4.6}$$ ●

If $E = I$, this reduces to the known state-space result.

V. CONCLUSION

We have presented tests for reachability and observability in descriptor systems which depend on matrices constructed from the fundamental matrix. These matrices reduce to the reachability and observability matrices in the state-space case. The descriptor open-loop control and boundary-value reconstruction problems were solved. It was shown that the reachable and unobservable subspaces should be thought of as residing in R^{2n}, not R^n, where n is the dimension of the descriptor variable.

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REFERENCES

[1] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, pp. 312-321, June 1977.
[2] ——, "Time-invariant descriptor systems," *Automatica*, vol. 14, pp. 473-480, 1978.
[3] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, pp. 411-425, Nov. 1976.
[4] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, pp. 702-707, June 1981.
[5] F. L. Lewis, "Descriptor systems: Decomposition into forward and backward subsystems," *IEEE Trans. Automat. Contr.*, vol. AC-29, Jan. 1984.
[6] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," *IEEE Trans. Automat. Contr.*, pp. 139-147, Feb. 1981.
[7] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1974.
[8] H. H. Rosenbrock, *State-Space and Multivariable Theory*. New York: Wiley, 1970.
[9] N. J. Rose, "The Laurent expansion of a generalized resolvent with some applications," *SIAM J. Appl. Math. Anal.*, vol. 9, pp. 751-758, Aug. 1978.
[10] C. E. Langenhop, "Controllability and stabilization of regular singular linear systems with constant coefficients," Dept. Math., Southern Illinois Univ., Carbondale, Dec. 6, 1979.
[11] F. L. Lewis, "Ajoint matrix, Cayley-Hamilton theorem, and Fadeev's method for the matrix pencil (sE-A)," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, 1983, pp. 1282-1288.
[12] C. E. Langenhop, "The Laurent expansion for a nearly singular matrix," *Linear Algebra Appl.*, vol. 4, pp. 329-340, 1971.
[13] K.-T. Wong, "The eigenvalue problem $\lambda Tx + Sx$," *J. Diff. Eq.*, vol. 16, pp. 270-281, 1974.
[14] F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," presented at the SIAM Nat. Meet., Denver, CO, June 1983.
[15] L. Pandolfi, "Controllability and stablization for linear systems of algebraic and differential equations," *J. Optimiz. Theory Appl.*, vol. 30, pp. 601-620, Apr. 1980.
[16] D. Cobb, "Controllability, observability, and duality in singular systems," submitted for publication.
[17] E. N. Khasina, "Control of singular linear dynamic systems," *Automat. Remote Contr.*, vol. 43, pp. 448-455, Apr. 1982.
[18] G. C. Verghese and T. Kailhath, "Impulsive behavior in dynamical systems: Structure and significance," in *Proc. 4th Int. Symp. Math. Theory, Networks Syst.*, Delft, The Netherlands, July 1979, pp. 162-168.
[19] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 111-129, Feb. 1981.
[20] S. L. Cambell, *Singular Systems of Differential Equations*. San Francisco, CA: Pitman, 1980.
[21] F. L. Lewis, "Descriptor systems: Fundamental matrix, reachability and observability matrices, subspaces," in *Proc. 23rd IEEE Conf. Decision Contr.*, Dec. 1984.
[22] D. G. Luenberger, "Nonlinear descriptor systems," *J. Econ. Dyn. Contr.*, vol. 1, pp. 219-242, 1979.
[23] S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*. London, England: Pitman, 1979.
[24] J. D. Aplevich, "Time domain input-output representations of linear systems," *Automatica*, vol. 17, no. 3, pp. 509-522, 1981.

A General Stability Theorem for Linear Discrete-Time Systems

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***Abstract*—The strongest sufficient conditions for constraining zeros to be on or within a circular region in the z plane in terms of weighted absolute norms are derived using elementary geometry. It is shown that the weighted L_1 norm yields the strongest result.**

I. INTRODUCTION

Recently, Dabke [1], Mori [2], and Berger [3], [4] have derived sufficient conditions for the stability of linear discrete-time systems involving the coefficients of the characteristic polynomial. It will be shown that the results are just special cases of a general theorem for constraining zeros to be on or within a circular region in the z plane of radius α using weighted absolute norms.

If the characteristic polynomial equation of a discrete-time system is

$$P(z) = z^n + t_1 z^{n-1} + \cdots t_n = 0 \tag{1}$$

the real vector $t^T = [t_1 \cdots t_n]$ can be represented by a point in n-

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dimensional Euclidean space. The domain of t (parameter space) corresponding to a circular region of radius α in the z plane was derived by Fam and Meditch [5]. This domain has the following properties:

1) it contains the origin
2) it is bounded by three surfaces.

The three surfaces are as follows.

1) An $(n - 1)$-dimensional hyperplane B_1, which corresponds to the real positive zero $z = \alpha$, i.e., $P(\alpha) = 0$ or

$$t_1^T \cdot g_1 = -\alpha^n \tag{3}$$

where g_1 is the n-dimensional vector

$$g_1^T = [\alpha^{n-1} : \cdots : \alpha : 1]. \tag{4}$$

2) An $(n - 1)$-dimensional hyperplane B_2, which corresponds to the real negative zero $z = -\alpha$, i.e., $P(-\alpha) = 0$ or

$$t_2^T \cdot g_2 = (-1)^{n-1}\alpha^n \tag{5}$$

where g_2 is the n-dimensional vector

$$g_2^T = [\cdots \; -\alpha^3 : \alpha^2 : \alpha^2 : -\alpha : 1]. \tag{6}$$

3) An $(n - 2)$-dimensional hyperplane B_3, which corresponds to the complex zero pair on the circle of radius α in the z-plane.

II. MAIN RESULTS

We will first establish a geometrical result for the largest body containing polynomials with maximum absolute value zero less than or equal to α in the parameter space using weighted absolute norms.

Lemma 1: Let a point in n-dimensional Euclidean space be $t^T = [t_1 \cdots t_n]$. Then the point (t^*) where the weighted absolute norm on t_i, that is,

$$\sum_{i=1}^{n} |t_i|^q \cdot m_i; \qquad q>1;\; m_i>0, \tag{7}$$

is a minimum subject to

$$\sum_{i=1}^{n} |t_i| \cdot p_i = k; \qquad p_i>0;\; k>0 \tag{8}$$

is

$$t_i^* = \frac{c_i \cdot k}{r} \tag{9}$$

where

$$c_i = \left[\frac{p_i}{m_i}\right]^{1/q-1}$$

$$r = \sum_{j=1}^{n} c_j \cdot p_j.$$

Proof: This is a standard minimization problem:

$$\min_{t_i,\, i=1,\, n} \sum_{i=1}^{n} t_i^q \cdot m_i; \qquad q>1;\; m_i>0;\; t_i \ge 0 \tag{10}$$

subject to the constraint

$$\sum_{i=1}^{n} t_i \cdot p_i = k; \qquad p_i>0;\; k>0;\; t_i \ge 0. \tag{11}$$

Using the Lagrange multiplier λ, we have an unconstrained minimization problem for minimizing

$$C = \sum_{i=1}^{n} t_i^q \cdot m_i - \lambda t_i p_i. \tag{12}$$

This minimum is found by requiring

$$\frac{\partial C}{\partial t_i} = q t_i^{q-1} \cdot m_i - \lambda p_i = 0 \tag{13}$$

and

$$\frac{\partial^2 C}{\partial t_i^2} = q(q-1) t_i^{q-2} \cdot m_i > 0. \tag{14}$$

From (13),

$$t_i^* = \left[\frac{\lambda p_i}{m_i q}\right]^{1/q-1} \tag{15}$$

and substituting this t_i^* into (11), we have

$$\left[\frac{\lambda}{q}\right]^{1/q-1} \cdot r = k \tag{16}$$

or

$$\lambda = \left[\frac{k}{r}\right]^{q-1} \cdot q. \tag{17}$$

Finally, from (17) and (15), we have

$$t_i^* = \frac{c_i \cdot k}{r} \tag{18}$$

which is the required result.

Theorem 1: The roots of the polynomial equation

$$P(z) = z^n + t_1 z^{n-1} + \cdots t_n = 0 \tag{19}$$

lie within or on a circle of radius α centred on the origin of the z plane if

$$\left[\sum_{i=1}^{n} |t_i|^q \cdot m_i\right]^{1/q} \le b; \qquad q>1;\; q \text{ real} \tag{20}$$

where

$$b = \left[\sum_{i=1}^{n} |f_i|^q \cdot m_i\right]^{1/q}$$

$$f_i = \frac{d_i \cdot \alpha^n}{s}$$

$$d_i = \left[\frac{\alpha^{n-i}}{m_i}\right]^{1/q-1}$$

$$s = \sum_{j=1}^{n} d_j \cdot \alpha^{n-j}.$$

Proof: Dabke [1] and Mori [2] have derived the following sufficient condition for constraining zeros on or within a circle of radius α centered on the origin of the z plane:

$$\sum_{i=1}^{n} |t_i| \cdot \alpha^{n-i} \le \alpha^n. \tag{21}$$

The largest geometrical object that can be fitted on or within the convex body described by (21) is, in fact, the object described by (20). This is proved by using Lemma 1 for $p_i = \alpha^{n-i}$ and $k = \alpha^n$.

Fam and Meditch [5] have shown that $P(\alpha)$ and $P(-\alpha)$ are the boundaries of the parameter space domain (t) corresponding to the circular region of radius α in the z plane. Since $P(\alpha)$ and $P(-\alpha)$ support the geometrical object uniquely, it represents the largest geometrical object given by (20) containing polynomials with maximum absolute value zero less than or equal to α.

This completes the proof.

Corollary 1: The roots of the polynomial equation

$$P(z)=z^n+t_1z^{n-1}+\cdots t_n=0 \tag{22}$$

lie within or on a circle of radius α centered on the origin of the z plane if

$$\left[\sum_{i=1}^{n}|t_i|^q\cdot\alpha^{n-i}\right]^{1/q}\leqq b;\qquad q\geqq 1;\ q \text{ real} \tag{23}$$

where

$$b=\left[\sum_{i=1}^{n}|h|\cdot\alpha^{n-i}\right]^{1/q}$$

$$h=\left[\frac{\alpha^n}{g}\right]^q$$

$$g=\sum_{j=1}^{n}\alpha^{n-j}.$$

Proof: The special case of $q = 1$ is verified by comparison to (21). Using Theorem 1, we put $m_i = \alpha^{n-1}$ to obtain (23) for $q > 1$.

Corollary 2: The roots of the polynomial equation

$$P(z)=z^n+t_1z^{n-1}+\cdots t_n=0 \tag{24}$$

lie within or on the unit circle centered on the origin of the z plane if

$$\left[\sum_{i=1}^{n}|t_i|^q\right]^{1/q}\leqq n^{(1-q/q)};\qquad q\geqq;\ q \text{ real}. \tag{25}$$

Proof: Use Corollary 1 for the unit circle in the z plane.

Remark: The Euclidean norm ($q = 2$) was obtained by Berger [3], [4] using Lyapunov's second method.

Corollary 3: The strongest result of all absolute norms for the zeros to lie within or on a circle of radius α centered on the origin of the z plane is (21).

Proof: The object described by (21) can be seen as an n-dimensional convex body bounded by hyperplanes described by

$$t^T\cdot g_p=\alpha^n \tag{26}$$

where

$$t^T=[t_1\ \cdots\ t_n] \tag{27}$$

$$g_p^T=[\pm\alpha^{n-1}\ \cdots\ :\ \pm\alpha\ :\ \pm 1]. \tag{28}$$

But $P(\alpha)$ and $P(-\alpha)$ hyperplanes are also the boundaries of the parameter space domain (t) corresponding to a circular region of radius α in the z plane. This completes the proof.

III. AN EXAMPLE

Find a forward path gain K which stabilizes the discrete-time unity feedback closed-loop system with the open-loop plant transfer function

$$H(z)=\frac{0.5Kz}{(z-1)(z-0.5)}.$$

The closed-loop characteristic equation is

$$P(z)=z^2+(0.5K-1.5)z+0.5.$$

The sufficient stable range for K is given by Corollary 2($q = 1$):

$$|0.5K-1.5|+0.5<1,$$

that is, $2 < K < 4$.

The actual stable range for K is $0 < K < 6$.

IV. CONCLUSION

It has been shown that the strongest criterion of all absolute norms for constraining zeros on or within a circle centered on the origin in the z plane is the weighted L_1 norm. The result can be used to check the stability of linear discrete-time systems.

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REFERENCES

[1] K. P. Dabke, "A simple criterion for stability of linear discrete system," *Int. J. Contr.*, vol. 36, pp. 657–659, 1983.
[2] T. Mori, "Note on the absolute value of the roots of a polynomial," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 54–55, Jan. 1984.
[3] C. S. Berger, "A simple sufficient condition for the stability of linear discrete time systems," *Int. J. Contr.*, vol. 35, pp. 1073–1080, 1982.
[4] ——, "Proof of a certain conjecture on the stability of linear discrete systems," *Int. J. Contr.*, vol. 36, pp. 545–546, 1982.
[5] A. T. Fam and J. S. Meditch, "A canonical parameter space for linear systems design," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 454–458, June 1978.

Sampling of a System with a Time Delay

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Abstract—**This note deals with the problem of sampling a continuous-time system which contains a time delay. It is shown that the infinite-dimensional continuous-time system can be represented by a finite-dimensional sampled data system. It is shown that there are simple expressions for the sampled data state-space representations.**

I. INTRODUCTION

A continuous-time linear system with a time delay is an infinite-dimensional system. To model the delay it is necessary to store a function of time over a time interval equal to the time delay. The sampled data representation of such a system is, however, finite dimensional, if we are interested in the states at the sampling instants only. This is due to the sampling mechanism and to the fact that the input signal is assumed to be

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