Linear Control Systems
Lecture # 7
Controllability

Motivating Example: A rocket in vertical motion may be modeled by

$$\begin{aligned} \dot{h} &= v \\ m\dot{v} &= -mg + f \end{aligned}$$

$$h = \text{altitude}, \;\; v = \text{velocity}, \;\; m = \text{mass}, \;\; f = \text{thrust force}$$

$$x_1 = h, \;\; x_2 = v, \;\; u = f/m - g$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Can we find a (continuous) control $u(t)$ over the period $[t_0, t_f]$ to move the state of the system from a given initial state $x(t_0) = x_0$ to a desired final state $x(t_f) = x_f$?

Definition: The system $\dot{x} = Ax + Bu$, or the pair (A, B), is said to be controllable on $[t_0, t_f]$ if given any initial state x_0, there is a continuous control $u(t)$ that steers the state of the system from $x(t_0) = x_0$ to $x(t_f) = 0$. It is said to be reachable on $[t_0, t_f]$ if given any final state x_f, there is a continuous control $u(t)$ that steers the state of the system from $x(t_0) = 0$ to $x(t_f) = x_f$

Remark: For time-invariant systems, we can, without loss of generality, take $t_0 = 0$

Mathematical preliminaries: Let M by an $m \times n$ matrix

$$Mx = 0 \text{ for } x \neq 0 \;\Rightarrow\; \text{rank } M < n$$

$$y^T M = 0 \text{ for } y \neq 0 \;\Rightarrow\; \text{rank } M < m$$

If M is $n \times n$, then

$$M \text{ is nonsingular} \Leftrightarrow \text{rank } M = n$$

For a nonsingular matrix M

$$Mx = 0 \Leftrightarrow x = 0$$

$$x^T M = 0 \Leftrightarrow x = 0$$

A symmetric $n \times n$ matrix P is positive semidefinite if

$$x^T P x \geq 0, \quad \forall\, x$$

It is positive definite if

$$x^T P x > 0, \quad \forall\, x \neq 0$$

P is positive definite

$$\Leftrightarrow \quad \text{All eigenvalues of } P \text{ are positive}$$

$$\Leftrightarrow \quad \text{All leading principal minors of } P \text{ are positive}$$

A symmetric positive semidefinite matrix is positive definite if and only if it is nonsingular

Controllability

The Controllability Gramian of (A, B) is defined by

$$W_c(0, t_f) = \int_0^{t_f} e^{-At} B B^T e^{-A^T t} \, dt$$

$W_c(0, t_f)$ is symmetric. It is positive semidefinite because

$$x^T W_c(0, t_f) x = \int_0^{t_f} x^T e^{-At} B B^T e^{-A^T t} x \, dt \geq 0$$

for all vectors x.

Lemma: $W_c(0, t_f)$ is positive definite if and only if there is no vector $x_a \neq 0$ such that

$$x_a^T e^{-At} B \equiv 0, \quad \forall\, t \in [0, t_f]$$

Proof:

$$x_a^T W_c(0, t_f) x_a = \int_0^{t_f} x_a^T e^{-At} B B^T e^{-A^T t} x_a \, dt$$

$$x_a^T W_c(0, t_f) x_a = 0 \;\; \Leftrightarrow \;\; x_a^T e^{-At} B \equiv 0, \; \forall\, t \in [0, t_f]$$

Theorem: The pair (A, B) is controllable (reachable) on $[0, t_f]$ if and only if the controllability Gramian $W_c(0, t_f)$ is positive definite

Proof of sufficiency: Suppose $W_c(0, t_f)$ is positive definite and let x_0 be the initial state and x_f be the desired final state

$$x(t_f) = e^{At_f}x(0) + \int_0^{t_f} e^{A(t_f-\tau)}Bu(\tau)\, d\tau$$

Take

$$u(t) = B^T e^{-A^T t} W_c^{-1}(0, t_f)v$$

for some constant vector v.

$$x(t_f) = e^{At_f}x_0 + \int_0^{t_f} e^{A(t_f-\tau)}BB^T e^{-A^T\tau} W_c^{-1}(0, t_f)v\, d\tau$$

$$x(t_f) = e^{At_f}x_0 + e^{At_f}\int_0^{t_f} e^{-A\tau}BB^Te^{-A^T\tau}\,d\tau\, W_c^{-1}(0,t_f)v$$

$$x(t_f) = e^{At_f}x_0 + e^{At_f}W_c(0,t_f)W_c^{-1}(0,t_f)v$$

$$x(t_f) = e^{At_f}x_0 + e^{At_f}v = e^{At_f}(x_0 + v)$$

$$v = -x_0 + e^{-At_f}x_f \quad \Rightarrow \quad x(t_f) = x_f$$

Thus,

$$u(t) = B^Te^{-A^Tt}W_c^{-1}(0,t_f)\left[-x_0 + e^{-At_f}x_f\right]$$

steers $x(0) = x_0$ to $x(t_f) = x_f$ for any x_0 and x_f

Proof of Necessity: We want to show that positive definiteness of $W_c(0, t_f)$ is a necessary condition for controllability (reachability) over $[0, t_f]$

We will us a contradiction argument. Suppose $W_c(0, t_f)$ is not positive definite. Then, there is a vector $x_a \neq 0$ such that

$$x_a^T e^{-At_f} B \equiv 0, \quad \forall\, t \in [0, t_f]$$

For the case of controllability, take $x_0 = x_a$ and suppose there is $u(t)$ such that $x(t_f) = 0$. Then,

$$0 = e^{At_f} x_a + \int_0^{t_f} e^{A(t_f - \tau)} B u(\tau)\, d\tau$$

Multiply from the left by $x_a^T e^{-At_f}$

$$0 = x_a^T e^{-At_f} e^{At_f} x_a + \int_0^{t_f} x_a^T e^{-At_f} e^{A(t_f-\tau)} Bu(\tau)\, d\tau$$

$$0 = x_a^T x_a + \int_0^{t_f} x_a^T e^{-A\tau} Bu(\tau)\, d\tau$$

$$0 = x_a^T x_a \quad \Rightarrow \quad x_a = 0 \ \ \text{Contradcition}$$

For the case of reachability, take $x_f = e^{At_f}x_a$ and suppose there is $u(t)$ that steers $x(0) = 0$ to $x(t_f) = x_f$. Then,

$$e^{At_f}x_a = \int_0^{t_f} e^{A(t_f-\tau)}Bu(\tau)\,d\tau$$

Multiply from the left by $x_a^T e^{-At_f}$

$$x_a^T e^{-At_f}e^{At_f}x_a = \int_0^{t_f} x_a^T e^{-At_f}e^{A(t_f-\tau)}Bu(\tau)\,d\tau$$

$$x_a^T x_a = \int_0^{t_f} x_a^T e^{-A\tau}Bu(\tau)\,d\tau$$

$$x_a^T x_a = 0 \;\;\Rightarrow\;\; x_a = 0 \;\; \text{Contradcition}$$

Rocket Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = Ax + Bu$$

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{-At}B = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

$$W_c(0, t_f) = \int_0^{t_f} \begin{bmatrix} -t \\ 1 \end{bmatrix} \begin{bmatrix} -t & 1 \end{bmatrix} dt$$

$$W_c(0,t_f) = \int_0^{t_f} \begin{bmatrix} t^2 & -t \\ -t & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t_f^3 & -\frac{1}{2}t_f^2 \\ -\frac{1}{2}t_f^2 & t_f \end{bmatrix}$$

Leading principal minors:

$$\frac{1}{3}t_f^3 > 0$$

$$\frac{1}{3}t_f^4 - \frac{1}{4}t_f^4 > 0$$

The system is controllable (reachable) on any interval $[0, t_f]$ with $t_f > 0$

Lemma: The controllability Gramian $W_c(0, t_f)$ is positive definite if and only if $\mathrm{rank}\ \mathcal{C} = n$, where

$$\mathcal{C} = [B, AB,\ A^2B,\ \ldots,\ A^{n-1}B]$$

is the controllability matrix ($\mathcal{C}$ is $n \times nm$ when A is $n \times n$ and B is $n \times m$)

Proof:

$W_c(0, t_f)$ is singular if and only if

$$\text{there is } x_a \neq 0 \text{ such that } x_a^T e^{-At} B \equiv 0,\ \forall\, t \in [0, t_f]$$

$$\Leftrightarrow \quad x_a^T \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A^k B \equiv 0,\ \forall\, t \in [0, t_f]$$

$$\Leftrightarrow \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} x_a^T A^k B \equiv 0, \ \forall\, t \in [0, t_f]$$

$$\Leftrightarrow \ x_a^T A^k B = 0, \ \text{for } k = 0, 1, 2, \ldots$$

By Cayley-Hamilton theorem, A satisfies its characteristic equation

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_n = 0$$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_n I = 0$$

Hence, A^k, for $k \geq n$, are linear combinations of $I, \ A, \ \ldots, \ A^{n-1}$

$$x_a^T A^k B = 0,\ \forall\, k \geq 0 \Leftrightarrow x_a^T A^k B = 0,\ \text{for}\ k = 0, 1, 2, \ldots, n$$

$W_c(0, t_f)$ is singular if and only if

$$x_a^T [B, AB,\ A^2B,\ \ldots,\ A^{n-1}B] = 0$$

$$\Leftrightarrow \quad \text{rank}\ [B, AB,\ A^2B,\ \ldots,\ A^{n-1}B] < n$$

Theorem: The pair (A, B) is controllable (reachable) if and only if $\text{rank}\ \mathcal{C} = n$

Remark: Controllability of (A, B) is independent of the interval $[0, t_f]$

Example: Investigate controllability of

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{C} = [B,\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{rank}\, \mathcal{C} = 2 \quad \Rightarrow \quad (A, B) \text{ is controllable}$$

Example: Investigate controllability of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{C} = [B,\ AB,\ A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

$$\text{rank}\,\mathcal{C} = 3 \quad \Rightarrow \quad (A, B) \text{ is controllable}$$

Matlab:

$$C = ctrb(A, B); rank(C)$$

Example: Investigate controllability of

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathcal{C} = [B,\ AB,\ A^2B] = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 \end{bmatrix}$$

$$\text{If } b_i = 0 \text{ for some } i,\ \text{rank}\, \mathcal{C} < 3$$

$$\text{If } \lambda_i = \lambda_j \text{ for some } i \neq j,\ \text{rank}\, \mathcal{C} < 3$$

$$\mathcal{C} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ 0 & (\lambda_2 - \lambda_1) b_2 & (\lambda_2^2 - \lambda_1^2) b_2 \\ 0 & (\lambda_3 - \lambda_1) b_3 & (\lambda_3^2 - \lambda_1^2) b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ 0 & (\lambda_2 - \lambda_1) b_2 & (\lambda_2^2 - \lambda_1^2) b_2 \\ 0 & 0 & * \end{bmatrix}$$

$$* = (\lambda_3^2 - \lambda_1^2) b_3 - (\lambda_2^2 - \lambda_1^2) \frac{(\lambda_3 - \lambda_1)}{(\lambda_2 - \lambda_1)} b_3$$

$$
\begin{aligned}
* &= (\lambda_3^2 - \lambda_1^2)b_3 - (\lambda_2^2 - \lambda_1^2)\frac{(\lambda_3 - \lambda_1)}{(\lambda_2 - \lambda_1)}b_3 \\
&= (\lambda_3 - \lambda_1)\left[(\lambda_3 + \lambda_1) - (\lambda_2 + \lambda_1)\right]b_3 \\
&= (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)b_3
\end{aligned}
$$

$$
\mathcal{C} \longrightarrow \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ 0 & (\lambda_2 - \lambda_1)b_2 & (\lambda_2^2 - \lambda_1^2)b_2 \\ 0 & 0 & (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)b_3 \end{bmatrix}
$$

The pair (A, B) is controllable if and only if $b_i \neq 0$ for all i and $\lambda_i \neq \lambda_j$ for all $i \neq j$