Lyapunov Stability Theory

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1 Introduction

In this lecture we consider the stability of equilibrium points of autonomous nonlinear systems, both in continuous and discrete time. Lyapunov stability theory is a standard tool and one of the most important tools in the analysis of nonlinear systems. It may be extended relatively easily to cover non-autonomous systems and to provide a strategy for constructing stabilizing feedback controllers. In the sequel we present the results for time-invariant systems. They may be derived for time varying systems as well, but the essential idea is more accessible for the time invariant case. For further reading you can consult [1].

2 Stability of Autonomous Systems

Consider the nonlinear autonomous (no forcing input) system

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathcal{D} \longrightarrow \mathbb{R}^n$ is a locally Lipschitz map from the domain $\mathcal{D} \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that the system (1) has an equilibrium point $\bar{x} \in \mathcal{D}$, i.e., $f(\bar{x}) = 0$. We would like to characterize if the equilibrium point $\bar{x}$ is stable. In the sequel, we assume that $\bar{x}$ is the origin of state space. This can be done without any loss of generality since we can always apply a change of variables to $\xi = x - \bar{x}$ to obtain

$$\dot{\xi} = f(\xi + \bar{x}) \triangleq g(\xi) \tag{2}$$

and then study the stability of the new system with respect to $\xi = 0$, the origin. We have the following two types of stability.

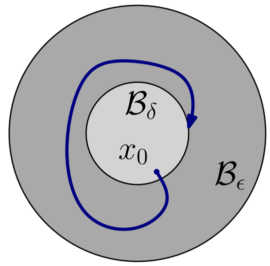

Figure 1: Illustration of a stable system

Definition 1 *The equilibrium $x = 0$ of (1) is*

1. ***stable***, *if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t > t_0 \tag{3}$$

2. ***asymptotically stable*** *if it is stable and in addition δ can be chosen such that*

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0. \tag{4}$$

3 Lyapunov's Direct Method

Let $V : \mathcal{D} \longrightarrow \mathbb{R}$ be a continuously differentiable function defined on the domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin. The rate of change of V along the trajectories of (1) is given by

$$\begin{aligned} \dot{V}(x) &\triangleq \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{d}{dt}x_i \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \dot{x} \quad = \frac{\partial V}{\partial x} f(x) \end{aligned} \tag{5}$$

The main idea of Lyapunov's theory is that if $\dot{V}(x)$ is negative along the trajectories of the system, then $V(x)$ will decrease as time goes forward. Moreover, we do not really need to solve the nonlinear ODE (1) for every initial condition, but we just need some information about the drift $f(x)$.

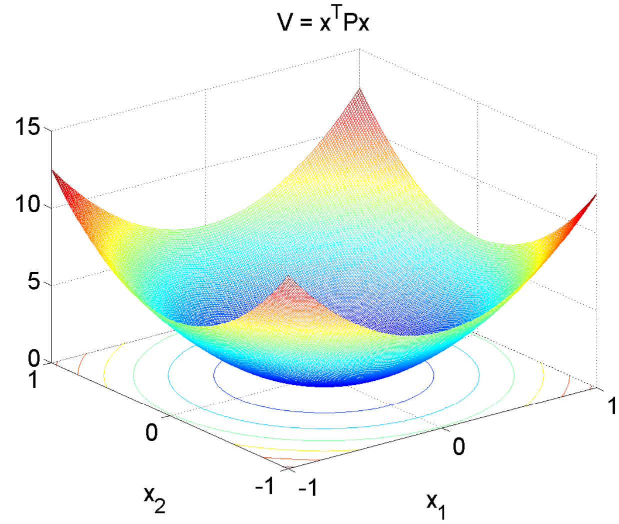

Figure 2: Lyapunov function in two states $V = x_1^2 + 1.5x_2^2$. The level sets are shown in the x_1x_2-plane.

Example 1 *Consider the nonlinear system*

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^2 x_2 \\ -x_2 \end{bmatrix}$$

and the candidate Lyapunov function

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

with $\lambda_1,\ \lambda_2 > 0$. If we plot the function $V(x)$ for some choice of λ's we obtain the result in Figure 2. This function has a unique minimum over all the state space at the origin. Moreover, $V(x) \to \infty$ as $\|x\| \to \infty$.

Calculate the derivative of V along the trajectories of the system

$$\dot{V}(x) = 2\lambda_1 x_1(-x_1 + 2x_1^2 x_2) + 2\lambda_2 x_2(-x_2) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$

Therefore, if $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$.

We are now ready to state Lyapunov's stability theorem.

Theorem 1 *Let the origin $x = 0 \in \mathcal{D} \subset \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} &V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \in D \setminus \{0\} \\ &\dot{V}(x) \leq 0, \quad \forall x \in D \end{aligned} \tag{6}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0, \quad \forall x \in D \setminus \{0\}$$

then $x = 0$ is asymptotically stable

Remark 1 *If $V(x) > 0$, $\forall x \in D \setminus \{0\}$, then V is called locally positive definite. If $V(x) \geq 0$, $\forall x \in D \setminus \{0\}$, then V is locally positive semi-definite. If the conditions (6) are met, then V is called a Lyapunov function for the system $\dot{x} = f(x)$.*

Proof: Given any $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{x \in \mathbb{R}^n, \|x\| \leq r\} \subset D$. Let $\alpha = \min\limits_{\|x\|=r} V(x)$. Choose $\beta \in (0, \alpha)$ and define $\Omega_\beta = \{x \in B_r, V(x) \leq \beta\}$.

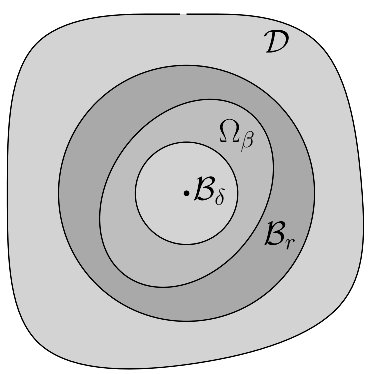

Figure 3: Various domains in the proof of Theorem 1

It holds that if $x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta$ $\forall t$ because

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta$$

Further $\exists \delta > 0$ such that $\|x\| < \delta \Rightarrow V(x) < \beta$. Therefore, we have that

$$B_\delta \subset \Omega_\beta \subset B_r$$

and furthermore

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Finally, it follows that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \ \forall t > 0$$

In order to show asymptotic stability, we need to to show that $x(t) \to 0$ as $t \to \infty$. In this case, it turns out that it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since V is monotonically decreasing and bounded from below by 0, then

$$V(x) \to c \geq 0, \text{ as } t \to \infty$$

Finally, it can be further shown by contradiction that the limit c is actually equal to 0.

$\square$

Example 2 *Recall Example 1. The derivative of the Lyapunov function candidate was given by*

$$\dot{V}(x) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$

For simplicity, assume that $\lambda_1 = \lambda_2 = 1$. Then

$$\dot{V}(x) = -2x_2^2 - 2x_1^2 g(x)$$

where $g(x) \triangleq 1 - 2x_1x_2$. Then the the derivative of V is guaranteed to be negative whenever $g(x) > 0$. The level sets of V, where $\dot{V} < 0$ will be invariant, or equivalently when $g(x) > 0 \Leftrightarrow x_1x_2 < 1/2$. So we conclude that the origin is locally asymptotically stable.

Example 3 *Consider a mass M connected to a spring, as shown in Figure 4, where $x = 0$ is defined as the equilibrium, the point where there is no force exerted by the spring, i.e.,*

$$F(x_1)x_1 > 0, \ \forall x_1 \neq 0, \quad F(x_1) = 0 \Leftrightarrow x_1 = 0$$

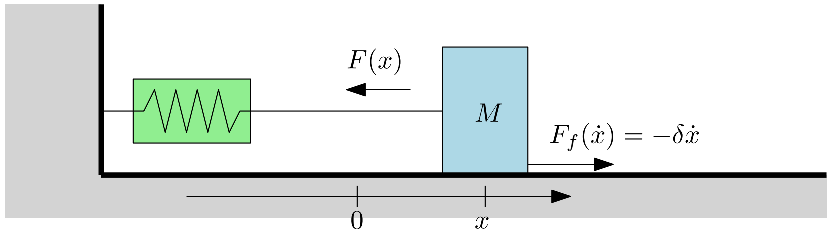

Figure 4: Mass spring system

The dynamics of this system are given by

$$M\ddot{x} = -F(x) - \delta\dot{x} \tag{7}$$

Let $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$ *and* $M = 1$, *then*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -F(x_1) - \delta x_2 \end{bmatrix} = f(x) \tag{8}$$

Consider the Lyapunov function candidate

$$V(x) = \int_0^{x_1} F(s)\,ds + \frac{1}{2}x_2^2$$

Then $\frac{\partial V(x)}{\partial x} = \begin{bmatrix} F(x_1), & x_2 \end{bmatrix}$ *and*[1]

$$\dot{V}(x) = F(x_1)\,x_2 + x_2\left(-F(x_1) - \delta x_2\right) = -\delta x_2^2 \leq 0$$

Therefore, the system is stable but we can't prove asymptotically stability because $\dot{V}\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right) = 0, \quad \forall x_1$. *However, we shall see that using LaSalle's invariance principle we can conclude that the system is asymptotically stable.*

The result in Theorem 1 can be extended to become a global result.

[1]Here we use the Leibnitz rule for differentiating integrals, i.e.,

$$\frac{d}{d\phi}\left(\int\limits_{a(\phi)}^{b(\phi)} f(\xi,\phi)\right)d\xi = \int\limits_{a(\phi)}^{b(\phi)} \frac{\partial}{\partial\phi} f(\xi,\phi)d\xi + f(b(\phi),\phi)\frac{d}{d\phi}b(\phi) - f(a(\phi),\phi)\frac{d}{d\phi}a(\phi)$$

Theorem 2 *Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \tag{9}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{10}$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0 \tag{11}$$

then the origin is globally asymptotically stable.

Remark 2 *if the function V satisfies the condition* (10)*, then it is said to be radially unbounded.*

Example 4 *Consider the system*

$$\dot{x} = \begin{bmatrix} x_2 \\ -h(x_1) - x_2 \end{bmatrix}$$

where the function h is locally Lipschitz with $h(0) = 0$ and $x_1 h(x_1) > 0, \ \forall x_1 \neq 0$. Take the Lyapunov function candidate

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(s) ds$$

The function V is positive definite for all $x \in \mathbb{R}^2$ and is radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = -(1-k)x_2^2 - k x_1 h(x_1) < 0$$

Hence the derivative of V is negative definite for all $x \in \mathbb{R}^2$, since $0 < k < 1$ (otherwise V is not positive definite! - convince yourself). Therefore, the origin is globally asymptotically stable.

4 Lyapunov's Indirect Method

In this section we prove stability of the system by considering the properties of the linearization of the system. Before proving the main result, we require an intermediate result.

Definition 2 *A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz or asymptotically stable, if and only if*

$$Re(\lambda_i) < 0, \forall i = 1, 2, \cdots, n$$

where λ_i's are the eigenvalues of the matrix F.

Consider the system $\dot{x} = Ax$. We look for a quadratic function

$$V(x) = x^T P x$$

where $P = P^T > 0$. Then

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T \left(A^T P + PA\right) x = -x^T Q x$$

If there exists $Q = Q^T > 0$ such that

$$A^T P + PA = -Q,$$

then V is a Lyapunov function and $x = 0$ is globally stable. This equation is called the *Matrix Lyapunov Equation.*

We formulate this as a matrix problem: Given Q positive definite, symmetric, how can we find out if there exists $P = P^T > 0$ satisfying the Matrix Lyapunov equation.

The following result gives existence of a solution to the Lypunov matrix equation for any given Q.

Theorem 3 *For $A \in \mathbb{R}^{n\times n}$ the following statements are equivalents:*

1. *A is Hurwitz*
2. *For all $Q = Q^T > 0$ there exists a unique $P = P^T > 0$ satisfying the Lyapunov equation*
$$A^T P + PA = -Q$$

Proof: We make a constructive proof of 1. $\Rightarrow$ 2. For a given $Q = Q^T > 0$, consider the following candidate solution for P:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

That $P = P^T > 0$ follows from the properties of Q. Note that the integral will converge if and only if A is a Hurwitz matrix. We now show that P satisfies the matrix Lyapunov equation:

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left[A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right] dt \\ &= \int_0^\infty \frac{d}{dt} \left[e^{A^T t} Q e^{At} \right] dt \\ &= e^{A^T t} Q e^{At} \Big|_0^\infty = -Q \end{aligned}$$

Thus P satisfies the matrix Lyapunov equation. In order to show uniqueness, assume that here exists another matrix $\bar{P} = \bar{P}^T > 0$ that solves the Lyapunov equation and $\bar{P} \neq P$. Then,

$$A^T(P - \bar{P}) + (P - \bar{P})A = 0$$

From which it follows that

$$0 = e^{A^T t} \left[A^T(P - \bar{P}) + (P - \bar{P})A \right] e^{At} = \frac{d}{dt} \left[e^{A^T t}(P - \bar{P}) e^{At} \right]$$

Therefore,

$$e^{A^T t}(P - \bar{P}) e^{At} = a, \forall t$$

where a is some constant matrix. Now, this also holds for $t = 0$, i.e.,

$$e^{A^T 0}(P - \bar{P}) e^{A0} = (P - \bar{P}) = e^{A^T t}(P - \bar{P}) e^{At} \to 0, \text{ as } t \to \infty$$

Where the last limit follows from the fact that A is Hurwitz. Therefore, $P = \bar{P}$.

The fact that 2. $\Rightarrow$ 1. follows from taking $V(x) = x^T P x$.

$\square$

Theorem 3 has an interesting interpretation in terms of the energy available to a system. If we say that the energy dissipated at a particular point in phase space x is given by $q(x) = x^T Q x$ - meaning that a trajectory passing through x is loosing $q(x)$units of energy per unit time, then the equation $V(x) = x^T P x$, where P satisfies the matrix Lyapunov equation gives the total amount of energy that the system will dissipate before reaching the origin. Thus $V(x) = x^T P x$ measures the energy stored in the state x. We are now ready to state Lyapunov's indirect method of stability.

Theorem 4 *Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable and $\mathcal{D}$ is a neighborhood of the origin. Let*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \tag{12}$$

then

1. *The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A*

2. *The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A*

Proof: If A is Hurwitz, then there exists $P = P^T > 0$ so that $V(x) = x^T P x$ is a Lyapunov function of the linearized system. Let us use V as a candidate Lyapunov function for the nonlinear system

$$\dot{x} = f(x) = Ax + (f(x) - Ax) \triangleq Ax + g(x)$$

The derivative of V is given by

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f(x)^T P x = x^T P A x + x^T A^T P x + 2x^T P g(x) && (13)\\ &= -x^T Q x + 2x^T P g(x) && (14)\end{aligned}$$

The function $g(x)$ satisfies

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g(x)\|_2 < \gamma \|x\|_2, \quad \forall \|x\|_2 < r$$

As such,

$$\dot{V}(x) < -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall \|x\|_2 < r$$

But we have the following fact $x^T Q x \geq \lambda_{\min}(Q) \|x\|_2^2$, where $\lambda_{\min}$ indicates the minimal eigenvalue of Q. Therefore,

$$\dot{V}(x) < -(\lambda_{\min}(Q) - 2\gamma \|P\|_2) \|x\|_2^2, \quad \forall \|x\|_2 < r$$

and choosing $\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|_2}$ renders $\dot{V}(x)$ negative definite (locally). Hence, the origin of the nonlinear system is asymptotically stable. This proves point 1.

The proof of point 2 shall be omitted; however, it relies on the Instability results in Section 6. More details can be found in [1, Ch. 4].

$\square$

Theorem 4 does not say anything when $Re(\lambda_i) \leq 0 \ \forall i$ with $Re(\lambda_i) = 0$ for some i. In this case linearization fails to determine the stability of the equilibrium point, and further analysis is necessary. The multi-dimensional result which is relevant here is the *Center Manifold Theorem*. This theorem is beyond the scope of this course.

Example 5 *The system $\dot{x} = ax^3$, $a > 0$ has an unstable equilibrium point at $x = 0$. The same system is asymptotically stable at the origin for $a < 0$. In both cases the linearized system is given by $\dot{x} = 0$, and we cannot conclude anything from the linearization or Lyapunov's indirect method. However, if we use Lyapunov's direct method with $V(x) = \frac{1}{4}x^4$, then $\dot{V}(x) = ax^6$ and if $a < 0$ then the system is globally asymptotically stable.*

5 The Invariance Principle

Definition 3 *A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called invariant for the system $\dot{x} = f(x)$, if*

$$\forall x(t_0) \in D \Rightarrow x(t) \in D, \ \forall t \in \mathbb{R}$$

A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called positively invariant for the system $\dot{x} = f(x)$, if

$$\forall x(t_0) \in D \Rightarrow x(t) \in D, \ \forall t \geq t_0$$

Theorem 5 (LaSalle's Invariance Principle) *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

Remark 3 *Theorem 5 result can also be used to find limit cycles.*

An important result that follows from Theorem 5 is the following corollary.

Corollary 1 *Let $x = 0 \in \mathcal{D}$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite function on the domain $\mathcal{D}$, such that $\dot{V}(x) \leq 0, \ \forall x \in \mathcal{D}$. Let $S \triangleq \{x \in \mathcal{D} \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

Example 6 *Consider again the mass-spring example 3. Apply LaSalles invariance principle by noting that*

$$\dot{V}(x) = 0 \Leftrightarrow x_2 = 0$$

As such $E = \{x \mid x_2 = 0\}$. Now, if $x_2 = 0$, then from the dynamics we obtain $F(x_1) = 0$ and since $F(x_1) = 0 \Leftrightarrow x_1 = 0$, we conclude that $M = \{0\}$. Thus, trajectories must converge to the origin, and we have proven that the system is asymptotically stable.

6 Instability Theorem

Theorem 6 *Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0, \text{ and } V(x_0) > 0$$

for some x_0 with arbitrary small $\|x_0\|$. Define the set

$$U = \{x \in \mathcal{B}_r \mid V(x) > 0\}$$

and suppose that $\dot{V}(x) > 0$ in U. Then $x = 0$ is unstable.

Sketch of proof: First note that x_0 belongs to the interior of U. Moreover, we can show that the trajectory starting at x_0 must leave the set U and that the trajectory leaves the set U through the surface $\|x\|_2 = r$. Since this can happen for arbitrarily small $\|x_0\|$, thus the origin is unstable.

$\square$

Note that the requirements on $V(x)$ are not as strict on the requirements on a Lyapunov function.

Example 7 *The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ is shown in Figure 5. Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

then $\dot{V}(x) = x_1^2 + x_2^2 > 0$ for $x \in U$, proving that the system is unstable.

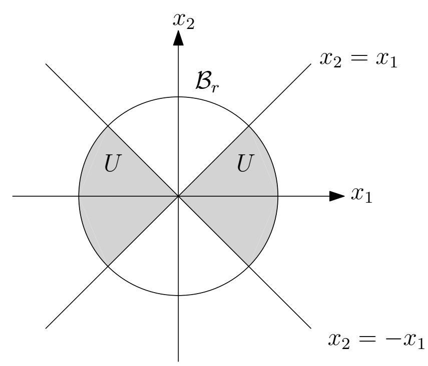

Figure 5: Instability regions

7 Stability of Discrete-Time Systems

Consider the discrete-time nonlinear system

$$x(k+1) = f(x(k)) \tag{15}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a nonlinear map, and we use the shorthand notation of indices k instead of the more precise one of kT_s, with T_s being the sampling period. We shall assume that the system has a equilibrium at the origin, i.e., $f(0) = 0$.

Remark 4 *For the discrete-time system* (15)*, the equilibrium point is characterized by the fixed point condition $x^* = f(x^*)$.*

Analogously to the continuous-time systems case, we can state the following global stability theorem.

Theorem 7 *Let the origin $x = 0 \in \mathbb{R}^n$ be an equilibrium point for the system* (15)*. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that*

$$\begin{aligned} V(0) = 0 \text{ and } V(x) > 0, \forall x \neq 0 \\ \Delta V(x(k)) \triangleq V(x(k)) - V(x(k-1)) < 0, \forall x(k) \in \mathcal{D} \\ \|x\| \to \infty \Rightarrow V(x) \to \infty \end{aligned} \tag{16}$$

then the origin is globally asymptotically stable.

Let us interpret the result in the previous theorem. For continuous time systems, we require that the derivative of the Lyapunov function is negative along the trajectories. For discrete-time systems, we require that the difference in the Lyapunov function is negative along the trajectories. Also, quite importantly we do not require that V is continuously differentiable, but to be only continuous.

If we now consider a linear discrete-time system given by

$$x(k+1) = Fx(k) \tag{17}$$

where $F \in \mathbb{R}^{n \times n}$. The asymptotic stability of such a system is characterized by the eigenvalues being strictly inside the unit circle in the complex plane.

Definition 4 *The matrix F is called Schur or asymptotically stable, if and only if*

$$|\lambda_i| < 1, \forall i = 1, \cdots, n$$

where λ_i's are the eigenvalues of the matrix F.

Theorem 8 *Consider the linear discrete-time system* (17)*, the following conditions are equivalent:*

1. *The matrix F is Schur stable*
2. *Given any matrix $Q = Q^T > 0$ there exists a positive definite matrix $P = P^T$ satisfying the discrete-time matrix Lyapunov equation*

$$F^T PF - P = -Q \tag{18}$$

Proof: Let's first show that 1. $\Rightarrow$ 2. Let F be Schur stable and take any matrix $Q = Q^T > 0$. Take the matrix $P = \sum_{i=0}^{\infty}(F^T)^i QF^i$, which is well defined by the asymptotic stability of F, and $P = P^T > 0$ by definition. Now, substitute P into (18)

$$\begin{aligned} F^T PF - P &= F^T \left(\sum_{i=0}^{\infty}(F^T)^i QF^i \right) F - \sum_{i=0}^{\infty}(F^T)^i QF^i \\ &= \sum_{i=1}^{\infty}(F^T)^i QF^i - \sum_{i=0}^{\infty}(F^T)^i QF^i = -Q \end{aligned}$$

In order to show uniqueness, suppose that there is another matrix $\bar{P}$ that satisfies the Lyapunov equation. After some reccurssions, we can show that if both P and $\bar{P}$ satisfy the Lyapunov equation then

$$(F^T)^N (P - \bar{P}) F^N = P - \bar{P}$$

Letting $N \to \infty$ yields the result.

In order to show that 2. $\Rightarrow$ 1., consider the Lyapunov function $V(x) = x^T P x$, and fix an initial state $x(0)$. We have that (by applying the Lyapunov equation recursively and summing up the steps)

$$V(x(N)) - V(x(0)) = -\sum_{i=0}^{N-1} x(i)^T Q x(i) \leq -\lambda_{min}(Q) \sum_{i=0}^{N-1} \|x(i)\|_2^2$$

Therefore, the sequence $[V(x(k))]_{k\in\mathbb{N}}$ is strictly decreasing and bounded from below, hence it attains a non-negative limit. We can further show by contradiction that this limit is actually 0, or equivalently $\lim_{i\to\infty} \|x(i)\| = 0$, since this holds for any choice of $x(0)$, it follows that F is Schur stable.

□

References

[1] H. K. Khalil. *Nonlinear systems.* Prentice hall, 3rd edition, 2002.