ECE 680: Linear System Review

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Today's Class

- Some remarks on state-space representation

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- Solving the state equation

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- Pole placement

Some Remarks on State-Space Representation

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- Signals are functions of time, which can be scalar-valued or vector-valued
- A system is any part of the real world surrounded by a well defined boundary
- The system is influenced by its environment via input signal, $\boldsymbol{u}(t)$ and acts on its environment via output signal $\boldsymbol{y}(t)$

State of the System

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- The state of the system contains all past information of the system up to the initial time t_0
- If we wish to compute the system output for $t > t_0$, we only need $\boldsymbol{u}(t)$ for $t > t_0$ and the initial state $x(t_0)$

Solving Uncontrolled State Equation

- Time-invariant linear model

$$\dot{x}(t) = Ax(t)$$

subject to an initial condition

$$x(0) = x_0$$

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- Solution, $\boxed{x(t) = e^{At}x_0}$

Solving Uncontrolled State Equation—More General Case

- $$\boxed{x(t) = e^{A(t-t_0)}x(t_0)}$$

Solving Uncontrolled State Equation—More General Case

- $$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0)}$$
- $e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{\Phi}(t, t_0)$
  State transition matrix—it relates the state at any instant of time t_0 to the state at any other time t

State Equation Solution of Controlled System

- Linear Time-Invariant (LTI) controlled dynamic system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State Equation Solution of Controlled System

- Linear Time-Invariant (LTI) controlled dynamic system

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

- Premultiply by $e^{-\boldsymbol{A}t}$

$$e^{-\boldsymbol{A}t}\dot{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{A}x(t) + e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

Solution of Controlled System

- Re-arrange

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

Solution of Controlled System

- Re-arrange

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

- Note that

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{x}(t)+e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t)$$

Solution of Controlled System

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$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

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$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t)$$

- Hence, $\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$

Controlled System Model Solution

- Integrate

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Controlled System Model Solution

- Integrate

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

- Manipulate to obtain

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Controlled System—General Case

Important Solution Formula

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Reachability Definition

We say that the system $\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is reachable if for any $\boldsymbol{x}_f$ there is $t_1 > 0$ and a control law, $\boldsymbol{u}(t)$, that transfers $\boldsymbol{x}(t_0) = \mathbf{0}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_f$

Controllability Definition

We say that the system $\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if there is a control law $\boldsymbol{u}(t)$ that transfers any initial state $x(t_0) = x_0$ to the origin at some time $t_1 > t_0$

- For **continuous** LTI systems controllability and reachability are equivalent

Some Controllability Tests

The following are equivalent:

- The system $\dot{x}(t) = Ax(t) + Bu(t)$ is reachable

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- $\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$

Some Controllability Tests

The following are equivalent:

- The system $\dot{x}(t) = Ax(t) + Bu(t)$ is reachable
- rank $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$
- The matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{-At} BB^\top e^{-A^\top t} dt$$

is nonsingular for all $t_1 > t_0$

Observability

- Suppose the system state is not directly accessible

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$$y = Cx + Du$$

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- Instead, we have the output of the system

$$y = Cx + Du$$

- We still want to know the behavior of the entire state

Observability Definition

The system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\}$$

or equivalently the pair (A, C), is observable if there is a finite $t_1 > t_0$ such that for arbitrary $u(t)$ and resulting $y(t)$ over $[t_0, t_1]$, we can determine $x(t_0)$ from complete knowledge of the system input u and output y

Remark on Observability

Note that once $x(t_0)$ is known, we can determine $x(t)$ from knowledge of $u(t)$ and $y(t)$ over any finite time interval $[t_0, t_1]$

Observability Test

The following are equivalent:

- The pair (A, C) is observable

Observability Test

The following are equivalent:

- The pair (A, C) is observable
- The observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n}$$

is of full rank n

Controller Design

- Plant (System to be controlled)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

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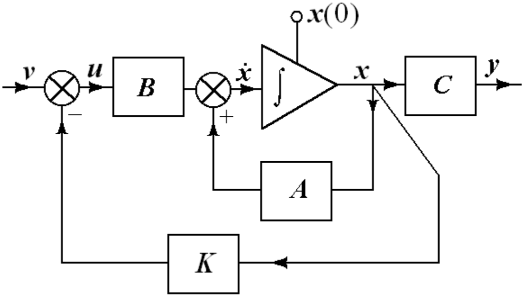
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

- Controller—linear state-feedback

$$u = -Kx + v$$

Closed-Loop System



$$\dot{x} = (A - BK)x + Bv$$