# EE 580 — Linear Control Systems

## *VI. State Transition Matrix*

Department of Electrical Engineering
Pennsylvania State University
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## 6.1 Introduction

Typical signal spaces are (infinite-dimensional) vector spaces. Consider the space $C(J, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued continuous functions on an interval $J = (a, b) \subset \mathbb{R}$. For $f$, $g \in C(J, \mathbb{R}^n)$, define $f + g \in C(J, \mathbb{R}^n)$ to be the function satisfying

$$(f + g)(t) = f(t) + g(t) \quad \text{for all} \quad t \in J.$$

Also, for $\alpha \in \mathbb{R}$ and $f \in C(J, \mathbb{R}^n)$, define $\alpha f \in C(J, \mathbb{R}^n)$ to be the function satisfying

$$(\alpha f)(t) = \alpha f(t) \quad \text{for all} \quad t \in J.$$

Then it is straightforward to show that $C(J, \mathbb{R}^n)$ is a vector space over the real field, where the origin $0 \in C(J, \mathbb{R}^n)$ is the function which is identically zero. Under this definition, a function $f \in C(J, \mathbb{R}^n)$ is a linear combination of functions $g_1, \ldots, g_m \in C(J, \mathbb{R}^n)$ if there are $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $f(t) = (\alpha_1 g_1 + \cdots + \alpha_m g_m)(t)$ for all $t \in J$. A nonempty finite subset $\{g_1, \ldots, g_m\}$ of $C(J, \mathbb{R}^n)$ is then linearly independent if and only if $(\alpha_1 g_1 + \cdots + \alpha_m g_m)(t) = 0$ for all $t \in J$ implies $\alpha_1 = \cdots = \alpha_m = 0$.

For simplicity, we will focus on LTI systems of the form $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$. However, all the theorems up to Section 6.3.2 inclusive carry over to LTV systems of the form $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ as long as $\mathbf{A}(\cdot)$ is continuous; see [1, Sections 2.3 & 2.4] and [2, Pages 50–73]. These results hold true even if $\mathbf{A}(\cdot)$ is piecewise continuous; a function is said to be *piecewise continuous* if, on each bounded interval in its domain, it is continuous except possibly for a finite number of discontinuities. If $\mathbf{A}(\cdot)$ is piecewise continuous (or piecewise constant in particular), then one can solve $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ piece by piece and then put together the results to obtain an overall solution.

## 6.2 Fundamental Matrices [3, Section 3.2.1]

### 6.2.1 Solution Space

Consider the homogeneous linear time-invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}, \tag{6.1}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a constant matrix. We have seen that this system, subject to any initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, has a unique solution $\boldsymbol{\phi}$ on $\mathbb{R}$; that is, there exists a unique $\boldsymbol{\phi} \in C(\mathbb{R}, \mathbb{R}^n)$ such that $\dot{\boldsymbol{\phi}}(t) = \mathbf{A}\boldsymbol{\phi}(t)$ for all $t \in \mathbb{R}$ and such that $\boldsymbol{\phi}(t_0) = \mathbf{x}_0$. It turns out that the set of solutions to (6.1) is a finite-dimensional subspace of $C(\mathbb{R}, \mathbb{R}^n)$, which is infinite dimensional.

**Theorem 6.1** *The set of all solutions to* (6.1) *forms an $n$-dimensional linear subspace of* $C(\mathbb{R}, \mathbb{R}^n)$.

*Proof.* Let $V$ be the space of all solutions of (6.1) on $\mathbb{R}$. It is easy to verify that $V$ is a linear subspace of $C(\mathbb{R}, \mathbb{R}^n)$. Choose a set of $n$ linearly independent vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ in $\mathbb{R}^n$. If $t_0 \in \mathbb{R}$, then there exist $n$ solutions $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n$ of (6.1) such that $\boldsymbol{\phi}_1(t_0) = \mathbf{x}_1, \dots, \boldsymbol{\phi}_n(t_0) = \mathbf{x}_n$. The linear independence of $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n\}$ follows from that of $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. It remains to show that $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n\}$ spans $V$. Let $\boldsymbol{\phi}$ be any solution of (6.1) on $\mathbb{R}$ such that $\boldsymbol{\phi}(t_0) = \mathbf{x}_0$. There exist unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\mathbf{x}_0 = \alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n$, and thus that $\boldsymbol{\psi} = \alpha_1 \boldsymbol{\phi}_1 + \dots + \alpha_n \boldsymbol{\phi}_n$ is a solution of (6.1) on $\mathbb{R}$ such that $\boldsymbol{\psi}(t_0) = \mathbf{x}_0$. However, because of the uniqueness of a solution satisfying the given initial data, we have that $\boldsymbol{\phi} = \alpha_1 \boldsymbol{\phi}_1 + \dots + \alpha_n \boldsymbol{\phi}_n$. This shows that every solution of (6.1) is a linear combination of $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n$, and hence that $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n\}$ is a basis of the solution space. □

### 6.2.2 Matrix Differential Equations

With $\mathbf{A} \in \mathbb{R}^{n \times n}$ as before, consider the *matrix differential equation*

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t). \tag{6.2}$$

If the columns of $\mathbf{X}(t) \in \mathbb{R}^{n \times n}$ are denoted $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t) \in \mathbb{R}^n$, then (6.2) is equivalent to

$$\begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \vdots \\ \dot{\mathbf{x}}_n(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \vdots \\ \mathbf{x}_n(t) \end{bmatrix},$$

so a matrix differential equation of the form (6.2) is a system of $n^2$ differential equations.

**Definition 6.2** *A set* $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n\} \subset C(\mathbb{R}, \mathbb{R}^n)$ *of $n$ linearly independent solutions of* (6.1) *is called a* fundamental set of solutions *of* (6.1), *and the $\mathbb{R}^{n \times n}$-valued function*

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_n \end{bmatrix}$$

*is called a* fundamental matrix *of* (6.1).

**Theorem 6.3** *A fundamental matrix $\boldsymbol{\Phi}$ of* (6.1) *satisfies the matrix equation* (6.2).

*Proof.* The result is immediate from

$$\dot{\boldsymbol{\Phi}}(t) = \begin{bmatrix} \dot{\boldsymbol{\phi}}_1(t) & \cdots & \dot{\boldsymbol{\phi}}_n(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}\boldsymbol{\phi}_1(t) & \cdots & \mathbf{A}\boldsymbol{\phi}_n(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \boldsymbol{\phi}_1(t) & \cdots & \boldsymbol{\phi}_n(t) \end{bmatrix} = \mathbf{A}\boldsymbol{\Phi}(t). \quad \square$$

Therefore, a fundamental matrix is a matrix-valued function whose columns span the space of solutions to the system of $n$ differential equations (6.1).

### 6.2.3 Characterizations of Fundamental Matrices

It can be shown that, if $\boldsymbol{\Phi}$ is a solution of the matrix differential equation (6.2), then

$$\det \boldsymbol{\Phi}(t) = e^{(\operatorname{tr} \mathbf{A})(t-\tau)} \det \boldsymbol{\Phi}(\tau) \tag{6.3}$$

for all $t, \tau \in \mathbb{R}$. This result is called *Abel's formula* and proved in [1, Theorem 2.3.3]. Its immediate consequence is that, since $t, \tau$ are arbitrary, we have either $\det \boldsymbol{\Phi}(t) = 0$ for all $t \in \mathbb{R}$ or $\det \boldsymbol{\Phi}(t) \neq 0$ for any $t \in \mathbb{R}$. In fact, we have the following characterizations of fundamental matrices, which we obtain without using Abel's formula.

**Theorem 6.4** *A solution* $\mathbf{\Phi}$ *of the matrix differential equation* (6.2) *is a fundamental matrix of* (6.1) *if and only if* $\mathbf{\Phi}(t)$ *is nonsingular for all* $t \in \mathbb{R}$.

*Proof.* To show necessity by contradiction, suppose that $\mathbf{\Phi} = [\boldsymbol{\phi}_1 \ \cdots \ \boldsymbol{\phi}_n]$ is a fundamental matrix for (6.1) and that $\det \mathbf{\Phi}(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then, since $\mathbf{\Phi}(t_0)$ is singular, the set $\{\boldsymbol{\phi}_1(t_0), \dots, \boldsymbol{\phi}_n(t_0)\} \subset \mathbb{R}^n$ is linearly dependent (over the real field), so that there are $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^n \alpha_i \boldsymbol{\phi}_i(t_0) = 0$. Every linear combination of the columns of a fundamental matrix is a solution of (6.1), and so $\sum_{i=1}^n \alpha_i \boldsymbol{\phi}_i$ is a solution of (6.1). Due to the uniqueness of the solution, this, along with the initial condition $\sum_{i=1}^n \alpha_i \boldsymbol{\phi}_i(t_0) = 0$, implies that $\sum_{i=1}^n \alpha_i \boldsymbol{\phi}_i$ is identically zero, which contradicts the fact that $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n$ are linearly independent. Thus, we conclude that $\det \mathbf{\Phi}(t) \neq 0$ for all $t \in \mathbb{R}$. To show sufficiency, let $\mathbf{\Phi}$ be a solution of (6.2) and suppose that $\det \mathbf{\Phi}(t) \neq 0$ for all $t \in \mathbb{R}$. Then the columns of $\mathbf{\Phi}$ form a linearly independent set of vectors for all $t \in \mathbb{R}$. Hence, $\mathbf{\Phi}$ is a fundamental matrix of (6.1). □.

For example, consider three $\mathbb{R}^{3\times 3}$-valued functions

$$\mathbf{\Phi}_1(t) = \begin{bmatrix} 1 & 2t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_2(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_3(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & t \end{bmatrix}, \quad t \in \mathbb{R}.$$

If we write $\mathbf{\Phi}_1 = [\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_3]$, where $\boldsymbol{\phi}_i \in C(\mathbb{R}, \mathbb{R}^3)$ are given by $\boldsymbol{\phi}_1(t) = [1 \ 0 \ 0]^{\mathrm{T}}$, $\boldsymbol{\phi}_2(t) = [2t \ 1 \ 0]^{\mathrm{T}}$, and $\boldsymbol{\phi}_3(t) = [t^2 \ t \ 1]^{\mathrm{T}}$ for all $t \in \mathbb{R}$, then $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\}$ is a linearly independent set in $C(\mathbb{R}, \mathbb{R}^3)$ because $\sum_{i=1}^3 \alpha_i \boldsymbol{\phi}_i$ being identically zero implies all $\alpha_i$ equal 0. Similarly, the columns of $\mathbf{\Phi}_2$ are linearly independent in $C(\mathbb{R}, \mathbb{R}^3)$ and so are those of $\mathbf{\Phi}_3$. Thus, $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$, and $\mathbf{\Phi}_3$ are "potentially" fundamental matrices. We have $\det \mathbf{\Phi}_1(t) = \det \mathbf{\Phi}_2(t) = 1$ and $\det \mathbf{\Phi}_3(t) = t$ for all $t \in \mathbb{R}$. Since $\det \mathbf{\Phi}_3(0) = 0$, however, Theorem 6.4 tells us that $\mathbf{\Phi}_3$ is not a fundamental matrix of (6.1) or even its time-varying version $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ (as the proof of Theorem 6.4 carries over to the time-varying case). On the other hand, since $\mathbf{\Phi}_1(t)$ and $\mathbf{\Phi}_2(t)$ are nonsingular for all $t$, it remains that $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are fundamental matrices provided that they solve the matrix differential equation (6.2). Solving $\dot{\mathbf{\Phi}}_1(t) = \mathbf{A}\mathbf{\Phi}_1(t)$ for $\mathbf{A} \in \mathbb{R}^{3\times 3}$ yields that

$$\mathbf{A} = \dot{\mathbf{\Phi}}_1(t)\mathbf{\Phi}_1(t)^{-1} = \begin{bmatrix} 0 & 2 & 2t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2t & t^2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so indeed $\mathbf{\Phi}_1$ is a fundamental matrix of the LTI system (6.1). However, since

$$\dot{\mathbf{\Phi}}_2(t)\mathbf{\Phi}_2(t)^{-1} = \begin{bmatrix} 0 & 1 & 2t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is a function of $t$, the matrix-valued function $\mathbf{\Phi}_2$ is not a fundamental matrix of (6.1). Nevertheless, $\mathbf{\Phi}_2$ is a fundamental matrix of the LTV system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, where $\mathbf{A}(t) = \dot{\mathbf{\Phi}}_2(t)\mathbf{\Phi}_2(t)^{-1}$.

**Theorem 6.5** *Let* $\mathbf{\Phi}$ *be a fundamental matrix of* (6.1). *Then* $\mathbf{\Psi}$ *is a fundamental matrix of* (6.1) *if and only if there exists a nonsingular matrix* $\mathbf{P} \in \mathbb{R}^{n\times n}$ *such that* $\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\mathbf{P}$ *for all* $t \in \mathbb{R}$.

*Proof.* Suppose $\mathbf{P}$ is invertible and $\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\mathbf{P}$ for all $t$. Then it is easy to check that $\mathbf{\Psi}$ is a solution of (6.2). Moreover, since $\det \mathbf{\Phi}(t) \neq 0$ for all $t$ and since $\det \mathbf{P} \neq 0$, we have $\det \mathbf{\Phi}(t)\mathbf{P} = \det \mathbf{\Phi}(t) \det \mathbf{P} \neq 0$ for all $t$. Thus by Theorem 6.4, $\mathbf{\Psi}$ is a fundamental matrix. Conversely, suppose that $\mathbf{\Psi}$ is a fundamental matrix. As $\mathbf{\Phi}(t)\mathbf{\Phi}(t)^{-1} = \mathbf{I}$ for all $t$, the chain rule gives

$$\tfrac{d}{dt}\big(\mathbf{\Phi}\mathbf{\Phi}^{-1}\big)(t) = \mathbf{0} \quad \Rightarrow \quad \tfrac{d}{dt}\big(\mathbf{\Phi}^{-1}\big)(t) = -\mathbf{\Phi}(t)^{-1}\tfrac{d}{dt}\big(\mathbf{\Phi}\big)(t)\mathbf{\Phi}(t)^{-1} = -\mathbf{\Phi}(t)^{-1}\mathbf{A}$$

for all $t \in \mathbb{R}$. Using this equality, we obtain

$$\tfrac{d}{dt}\big(\mathbf{\Phi}^{-1}\mathbf{\Psi}\big)(t) = \mathbf{\Phi}(t)^{-1}\tfrac{d}{dt}\big(\mathbf{\Psi}\big)(t) + \tfrac{d}{dt}\big(\mathbf{\Phi}^{-1}\big)(t)\mathbf{\Psi}(t) = \mathbf{\Phi}(t)^{-1}\mathbf{A}\mathbf{\Psi}(t) - \mathbf{\Phi}(t)^{-1}\mathbf{A}\mathbf{\Psi}(t) = \mathbf{0}$$

for all $t$. Hence $\mathbf{\Phi}(\cdot)^{-1}\mathbf{\Psi}(\cdot)$ is constant, which implies the existence of a matrix $\mathbf{P}$ such that $\mathbf{\Phi}(t)^{-1}\mathbf{\Psi}(t) = \mathbf{P}$ for all $t$; since $\mathbf{\Phi}(t)^{-1}$ and $\mathbf{\Psi}(t)$ are both nonsingular, $\mathbf{P}$ is nonsingular. □

## 6.3 State Transition Matrix [3, Sections 3.2.2 & 3.3.1]

### 6.3.1 Definitions

Solving the matrix differential equation (6.2) subject to the initial condition $\mathbf{X}(t_0) = \mathbf{I} \in \mathbb{R}^{n\times n}$ is equivalent to solving the initial-value problems

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}\mathbf{x}_i(t), \quad t \in \mathbb{R}; \quad \mathbf{x}_i(t_0) = \mathbf{e}_i \in \mathbb{R}^n, \tag{6.4}$$

separately over all $i = 1, \dots, n$, where $\mathbf{e}_i$ denotes the $i$th standard basis vector (i.e., $i$th column of the identity matrix $\mathbf{I}$). That is, if $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n \in C(\mathbb{R}, \mathbb{R}^n)$ are the unique solutions to (6.4) such that $\boldsymbol{\phi}_i(t_0) = \mathbf{e}_i$ and $\dot{\boldsymbol{\phi}}(t) = \mathbf{A}\boldsymbol{\phi}(t)$ for all $t \in \mathbb{R}$ and for each $i = 1, \dots, n$, then the fundamental matrix $\mathbf{\Phi}(\cdot, t_0) = [\boldsymbol{\phi}_1(\cdot) \ \cdots \ \boldsymbol{\phi}_n(\cdot)]$ is the unique solution to the matrix differential equation

$$\frac{\partial}{\partial t}\mathbf{\Phi}(t, t_0) = \mathbf{A}\mathbf{\Phi}(t, t_0)$$

subject to

$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I}.$$

The $\mathbb{R}^{n\times n}$-valued function $\mathbf{\Phi}(\cdot, t_0)$ is called the *state transition matrix* of (6.1). Every $\mathbf{x}_0 \in \mathbb{R}^n$ is a linear combination of the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ (i.e., $\mathbf{x}_0 = \mathbf{I}\mathbf{x}_0$), and so, as we already know, the solution to the system of differential equations (6.1) subject to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has a unique solution given by $\boldsymbol{\phi}(\cdot) = \mathbf{\Phi}(\cdot, t_0)\mathbf{x}_0$, which is the same linear combination of the columns of the state transition matrix. Moreover, $\mathbf{\Phi}(t, t_0)$ is determined by the Peano-Baker series, which in the case of LTI systems reduces to the *matrix exponential* given by

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k (t-t_0)^k}{k!}$$

for all $t$, $t_0 \in \mathbb{R}$ with the convention that $\mathbf{A}^0 = \mathbf{I}$ and $0! = 1$.

### 6.3.2 Properties of State Transition Matrix

**Theorem 6.6** *Let* $\mathbf{\Phi}(t, t_0)$ *be the state transition matrix of* (6.1). *If* $\mathbf{P}$ *is invertible, then the state variable change* $\mathbf{P}\bar{\mathbf{x}}(t) = \mathbf{x}(t)$ *leads to the* equivalent *state equation*

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}\bar{\mathbf{x}}(t), \quad t \in \mathbb{R}; \quad \bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{6.5}$$

*If* $\mathbf{\Phi}(\cdot, t_0)$ *is the state transition matrix of* (6.1), *then the state transition matrix of* (6.5) *is*

$$\bar{\mathbf{\Phi}}(t, t_0) = \mathbf{P}^{-1}\mathbf{\Phi}(t, t_0)\mathbf{P}, \quad t,\ t_0 \in \mathbb{R}. \tag{6.6}$$

*Proof.* Equation (6.5) follows from $\bar{\mathbf{x}}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$ and $\frac{d}{dt}\big(\mathbf{P}^{-1}\mathbf{x}\big)(t) = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{P}^{-1}\mathbf{x}(t)$, and equation (6.6) from $\frac{\partial}{\partial t}\big(\mathbf{P}^{-1}\mathbf{\Phi}(t, t_0)\mathbf{P}\big) = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{P}^{-1}\mathbf{\Phi}(t, t_0)\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{\Phi}(t_0, t_0)\mathbf{P} = \mathbf{I}$. □

**Theorem 6.7** *Let* $\mathbf{\Phi}(\cdot, t_0)$ *be the state transition matrix of* (6.1). *Then the following hold:*

*(a) If* $\mathbf{\Psi}$ *is any fundamental matrix of* (6.1), *then* $\mathbf{\Phi}(t, t_0) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}$ *for all* $t$, $t_0 \in \mathbb{R}$;

*(b)* $\mathbf{\Phi}(t, t_0)$ *is nonsingular and* $\mathbf{\Phi}(t, t_0)^{-1} = \mathbf{\Phi}(t_0, t)$ *for all* $t$, $t_0 \in \mathbb{R}$;

*(c)* $\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t, s)\mathbf{\Phi}(s, t_0)$ *for all* $t$, $s$, $t_0 \in \mathbb{R}$; *(semigroup property)*

*Proof.* Part (a) follows from $\frac{\partial}{\partial t}\big(\mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}\big) = \mathbf{A}\mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}$ and $\mathbf{\Psi}(t_0)\mathbf{\Psi}(t_0)^{-1} = \mathbf{I}$. Choose any fundamental matrix $\mathbf{\Psi}$. Then $\det \mathbf{\Psi}(t) \neq 0$ for all $t$, so (a) implies $\mathbf{\Phi}(t, t_0)$ is nonsingular for all $t$, $t_0$ because $\det \mathbf{\Phi}(t, t_0) = \det\big(\mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}\big) = \det \mathbf{\Psi}(t) \det \mathbf{\Psi}(t_0)^{-1} \neq 0$. Moreover, $\mathbf{\Phi}(t, t_0)^{-1} = \big(\mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}\big)^{-1} = \mathbf{\Psi}(t_0)\mathbf{\Psi}(t)^{-1}$, so (b) holds. Finally, for any fundamental matrix $\mathbf{\Psi}$, we have $\mathbf{\Phi}(t, t_0) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1} = \big(\mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1}\big)\big(\mathbf{\Psi}(s)\mathbf{\Psi}(t_0)^{-1}\big)$, so (c) holds. □

As an example, consider the homogeneous LTI system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

It follows readily from $\dot{x}_3(t) = 0$, $\dot{x}_2(t) = x_3(t)$, and $\dot{x}_1(t) = 2x_2(t)$ that $\boldsymbol{\phi}_1(t) = [1\ 0\ 0]^{\mathrm{T}}$, $\boldsymbol{\phi}_2(t) = [2t\ 1\ 0]^{\mathrm{T}}$, and $\boldsymbol{\phi}_3(t) = [t^2\ t\ 1]^{\mathrm{T}}$ are solutions of the system among infinitely many others. Moreover, $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\}$ is a linearly independent set in $C(\mathbb{R}, \mathbb{R}^3)$. Thus $\mathbf{\Psi} = [\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2\ \boldsymbol{\phi}_3]$ is a fundamental matrix of the system. Then by part (a) of Theorem 6.7 we have that the state transition matrix of the system is

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1} = \begin{bmatrix} 1 & 2t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2t_0 & t_0^2 \\ 0 & 1 & -t_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2(t-t_0) & (t-t_0)^2 \\ 0 & 1 & t-t_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Indeed, we have $\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$ for all $t$, $t_0 \in \mathbb{R}$. As another example, consider the homogeneous LTV system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

As $\boldsymbol{\phi}_1(t) = [1\ 0\ 0]^{\mathrm{T}}$, $\boldsymbol{\phi}_2(t) = [t\ 1\ 0]^{\mathrm{T}}$, and $\boldsymbol{\phi}_3(t) = [t^2\ t\ 1]^{\mathrm{T}}$ are linearly independent solutions to this LTV system, the matrix-valued function $\mathbf{\Psi} = [\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2\ \boldsymbol{\phi}_3]$ is a fundamental matrix of the LTV system (but not of any LTI system). Therefore, the state transition matrix of the LTV system is

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1} = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -t_0 & 0 \\ 0 & 1 & -t_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t-t_0 & t(t-t_0) \\ 0 & 1 & t-t_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

That is, $\frac{\partial}{\partial t}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$ for all $t \in \mathbb{R}$ with $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$.

### 6.3.3 Properties of Matrix Exponentials

**Theorem 6.8** *Let* $\mathbf{A} \in \mathbb{R}^{n \times n}$. *Then the following hold:*

*(a)* $e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = e^{\mathbf{A}(t_1+t_2)}$ *for all* $t_1$, $t_2 \in \mathbb{R}$;

*(b)* $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$ *for all* $t \in \mathbb{R}$*;*

*(c)* $\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}$ *for all* $t \in \mathbb{R}$*;*

*(d)* $\det e^{\mathbf{A}t} = e^{(\operatorname{tr}\mathbf{A})t}$ *for all* $t \in \mathbb{R}$*;*

*(e) If* $\mathbf{AB} = \mathbf{BA}$*, then* $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{(\mathbf{A}+\mathbf{B})t}$ *for all* $t \in \mathbb{R}$*.*

*Proof.* Part (a) holds because $e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = \mathbf{\Phi}(t_1, 0)\mathbf{\Phi}(0, -t_2) = \mathbf{\Phi}(t, -t_2) = e^{\mathbf{A}(t_1+t_2)}$, where the second equality follows from Theorem 6.7(c). Part (b) follows from

$$\mathbf{A}\left(\lim_{m\to\infty}\sum\nolimits_{k=0}^{m}\tfrac{\mathbf{A}^k t^k}{k!}\right) = \lim_{m\to\infty}\sum\nolimits_{k=0}^{m}\tfrac{\mathbf{A}^{k+1}t^k}{k!} = \left(\lim_{m\to\infty}\sum\nolimits_{k=0}^{m}\tfrac{\mathbf{A}^k t^k}{k!}\right)\mathbf{A}.$$

Part (c) is immediate from Theorem 6.7(b). If $\mathbf{\Psi}$ is a fundamental matrix of (6.1), then Theorem 6.7(a) gives $\det e^{\mathbf{A}t} = \det\left(\mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1}\right) = \det\mathbf{\Psi}(t)/\det\mathbf{\Psi}(0)$, where the last equality follows from $1 = \det\mathbf{I} = \det\left(\mathbf{\Psi}(0)\mathbf{\Psi}(0)^{-1}\right) = \det\mathbf{\Psi}(0)\det\mathbf{\Psi}(0)^{-1}$. Thus Abel's formula (6.3) gives (d). Finally, if $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{A}^i\mathbf{B}^j = \mathbf{B}^j\mathbf{A}^i$ and hence we have

$$\left(\lim_{m\to\infty}\sum\nolimits_{i=0}^{m}\tfrac{\mathbf{A}^i t^i}{i!}\right)\left(\lim_{l\to\infty}\sum\nolimits_{k=0}^{l}\tfrac{\mathbf{B}^j t^j}{j!}\right) = \left(\lim_{l\to\infty}\sum\nolimits_{k=0}^{l}\tfrac{\mathbf{B}^j t^j}{j!}\right)\left(\lim_{m\to\infty}\sum\nolimits_{i=0}^{m}\tfrac{\mathbf{A}^i t^i}{i!}\right),$$

which implies (e). □

In general, matrices do not commute. For example, if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

then $\mathbf{AB} \neq \mathbf{BA}$. In this case,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad e^{\mathbf{B}t} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \quad e^{(\mathbf{A}+\mathbf{B})t} = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix},$$

so $e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{(\mathbf{A}+\mathbf{B})t}$.

## 6.4 How to Determine Matrix Exponentials [3, Section 3.3.2]

We have already seen that the state transition matrix of a linear (time-invariant or time-varying) system can be obtained from a fundamental matrix, whose columns are linearly independent solutions to the given homogeneous system. Other methods to obtain the state transition matrix of an LTI system are summarized in this section.

### 6.4.1 Infinite Series Method

Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, evaluate the partial sum

$$\mathbf{S}_m(t) = \sum_{k=0}^{m}\frac{t^k}{k!}\mathbf{A}^k = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A} + \cdots + \frac{t^m}{m!}\mathbf{A}^m$$

for $m = 0, 1, \ldots$. Then, since $\mathbf{S}_m(t) \to e^{\mathbf{A}t}$ uniformly on any bounded interval $J$ in $\mathbb{R}$, it is guaranteed that $e^{\mathbf{A}t} \approx \mathbf{S}_m(t)$ for all $t \in J$ and for all sufficiently large $m$. If $\mathbf{A}$ is nilpotent, then the partial sum $\mathbf{S}_m(t)$, with $m = n - 1$, gives a closed-form expression for $e^{\mathbf{A}t}$. For example,

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad \mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} = \begin{bmatrix} 1 & \alpha t \\ 0 & 1 \end{bmatrix}.$$

### 6.4.2 Similarity Transformation Method

*Diagonalizable Case.* If $\mathbf{A} \in \mathbb{R}^{n\times n}$ has $n$ linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ corresponding to its (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_n$, then let $\mathbf{P} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$. Then, since $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, Theorem 6.6 gives

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}\mathbf{P}^{-1}.$$

*General Case.* Generate $n$ linearly independent generalized eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $\mathbf{P} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ takes $\mathbf{A}$ into the Jordan canonical form $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \operatorname{diag}\{\mathbf{J}_0, \dots, \mathbf{J}_m\} \in \mathbb{C}^{n\times n}$, where $\mathbf{J}_1, \dots, \mathbf{J}_m$ are Jordan blocks of varying dimensions. Then

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\mathbf{J}_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\mathbf{J}_m t}\end{bmatrix}\mathbf{P}^{-1}.$$

A Jordan block is the sum of a diagonal matrix and a nilpotent matrix; that is, $\mathbf{J}_k = \mathbf{\Lambda}_k + \mathbf{N}_k$ with

$$\mathbf{\Lambda}_k = \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \quad \text{and} \quad \mathbf{N}_k = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix},$$

where $\lambda_k$ is some eigenvalue of $\mathbf{A}$ for each $k$. Let $\mathbf{J}_k \in \mathbb{C}^{n_k\times n_k}$ for each $k = 1, \dots, m$, so that $\sum_{k=1}^m n_k = n$. Then we have $\mathbf{N}_k^{n_k} = \mathbf{0}$, and so the series defining $e^{\mathbf{N}_k t}$ terminates for each $k$. Moreover, direct computation yields $\mathbf{\Lambda}_k\mathbf{N}_k = \mathbf{N}_k\mathbf{\Lambda}_k$ for each $k$. Therefore, Theorem 6.8(e) yields

$$e^{\mathbf{J}_k t} = e^{\mathbf{\Lambda}_k t}e^{\mathbf{N}_k t} = e^{\lambda_k t}\begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{n_k-1}/(n_k-1)! \\ 0 & 1 & t & \cdots & t^{n_k-2}/(n_k-2)! \\ 0 & 0 & 1 & \cdots & t^{n_k-3}/(n_k-3)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad k = 1, \dots, m. \tag{6.7}$$

Note that, if $\lambda_k$ in (6.7) is complex, then *Euler's formula* gives

$$e^{\lambda_k t} = e^{(\sigma_k + i\omega_k)t} = e^{\sigma_k t}(\cos\omega_k t + i\sin\omega_k t).$$

For example, if $\alpha \neq 0$, then

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \quad\Rightarrow\quad \mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \quad\Rightarrow\quad e^{\mathbf{A}t} = \mathbf{P}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 \\ \alpha t & 1 \end{bmatrix}.$$

### 6.4.3 Cayley-Hamilton Theorem Method

In view of the Cayley-Hamilton Theorem, there exist $\beta_i(m;\cdot) \in C(\mathbb{R},\mathbb{R})$ for $i = 0, \dots, n-1$ and $m = 1, 2, \dots$ such that, for $\mathbf{A} \in \mathbb{R}^{n\times n}$,

$$e^{\mathbf{A}t} = \lim_{m\to\infty}\sum_{k=0}^{m}\frac{t^k}{k!}\mathbf{A}^k = \lim_{m\to\infty}\sum_{i=0}^{n-1}\beta_i(m;t)\mathbf{A}^i = \sum_{i=0}^{n-1}\lim_{m\to\infty}\beta_i(m;t)\mathbf{A}^i.$$

Then, letting $\beta_i(t) = \lim_{m\to\infty} \beta_i(m;t)$ for $i = 0, \ldots, n-1$ and $t \in \mathbb{R}$, we obtain

$$e^{\mathbf{A}t} = \sum_{i=0}^{n-1} \beta_i(t)\mathbf{A}^i, \quad t \in \mathbb{R}. \tag{6.8}$$

Thus one can determine $e^{\mathbf{A}t}$ by obtaining $\beta_i(t)$ for all $i$ and $t$. Let $f(s)$ and $g(s)$ be two analytic functions (i.e., functions locally given by a convergent power series; e.g., polynomials, exponential functions, trigonometric functions, logarithms, etc.). Let $p(s) = \prod_{j=1}^{p}(s-\lambda_j)^{m_k}$ be the characteristic polynomial of $\mathbf{A}$. Then $f(\mathbf{A}) = g(\mathbf{A})$ if

$$\frac{d^k f}{ds^k}(\lambda_j) = \frac{d^k g}{ds^k}(\lambda_j), \quad k = 0, \ldots, m_j - 1, \quad k = 1, \ldots, p, \tag{6.9}$$

where $\sum_{j=1}^{p} m_j = n$. (*Proof.* Equations (6.9) imply that $f(s) - g(s)$ has $p(s)$ as a factor, so the result follows from the Cayley-Hamilton Theorem.) Thus the terms $\beta_i(t)$ in (6.8) can be determined by letting

$$f(s) = e^{st} \quad \text{and} \quad g(s) = \beta_0(t) + \beta_1(t)s + \cdots + \beta_{n-1}(t)s^{n-1}.$$

For example, if $\mathbf{A} = \left[\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right]$, then the characteristic polynomial of $\mathbf{A}$ is $p(s) = \det(s\mathbf{I} - \mathbf{A}) = s^2$. Let $f(s) = e^{st}$ and $g(s) = \beta_0(t) + \beta_1(t)s$. Then, as $f(0) = g(0)$ and $\frac{\partial f}{\partial s}(0) = \frac{\partial g}{\partial s}(0)$ imply $\beta_0(t) = 1$ and $\beta_1(t) = t$, we obtain that $e^{\mathbf{A}t} = g(\mathbf{A}) = \mathbf{I} + t\mathbf{A}$.

### 6.4.4 Laplace Transform Method

We know that $e^{\mathbf{A}t}$ is the inverse Laplace transform of $(s\mathbf{I}-\mathbf{A})^{-1}$ for $\mathbf{A} \in \mathbb{R}^{n\times n}$. The partial fraction expansion of each entry in $(s\mathbf{I} - \mathbf{A})^{-1}$ gives

$$(s\mathbf{I} - \mathbf{A})^{-1} = \sum_{j=1}^{p} \sum_{k=0}^{m_j-1} \frac{k!}{(s-\lambda_j)^{k+1}} \mathbf{A}_{jk},$$

where $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of $\mathbf{A}$, with corresponding algebraic multiplicities $m_1, \ldots, m_p$ such that $\sum_{j=1}^{p} m_i = n$, and where each $\mathbf{A}_{jk} \in \mathbb{C}^{n\times n}$ is a matrix of partial fraction expansion coefficients. Taking the inverse Laplace transform gives

$$e^{\mathbf{A}t} = \sum_{j=1}^{p} \sum_{k=0}^{m_j-1} t^k e^{\lambda_j t} \mathbf{A}_{jk}. \tag{6.10}$$

If some eigenvalues are complex, conjugate terms on the right side of (6.10) can be combined using Euler's formula to give a real representation. If $\mathbf{A}_{jk}$ is nonzero, then $t^k e^{\lambda_j t}$ is called a *mode* of the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, $t \in \mathbb{R}$. It is easily seen that $\mathbf{x}(t) \to 0$ as $t \to \infty$ under arbitrary initial conditions if each mode of the system converges to zero (i.e., each $\lambda_j$ has negative real part). For example, if $\mathbf{A} = \left[\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right]$, then $e^{\mathbf{A}t}$ is the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1} = \left[\begin{smallmatrix} 1/s & 0 \\ \alpha/s^2 & 1/s \end{smallmatrix}\right]$. The modes of the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ in this example are 1 and $t$.

## 6.5 Discrete-Time Case

### 6.5.1 State Transition Matrix

Given an $\mathbf{A} \in \mathbb{R}^{n\times n}$, consider the discrete-time initial-value problem of solving

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \quad t = t_0,\ t_0 + 1,\ \ldots; \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{6.11}$$

The discrete-time state transition matrix is defined to be the unique solution $\mathbf{\Phi}(\cdot, t_0) \in \mathbb{R}^{n\times n}$ of the matrix difference equation

$$\mathbf{\Phi}(t+1, t_0) = \mathbf{A}\mathbf{\Phi}(t, t_0), \quad t = t_0,\ t_0+1,\ \ldots,$$

subject to

$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I}.$$

That is, $\mathbf{\Phi}(t, t_0) = \mathbf{A}^{t-t_0}$ for all $t$, $t_0 \in \mathbb{Z}$ with $t \geq t_0$, and as we have seen earlier the unique solution to (6.11) is given by $\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0$, $t \geq t_0$. Unlike the continuous-time case, the difference equation (6.11) cannot be solved backward in time unless $\mathbf{A}$ is nonsingular. This is because, unless $\mathbf{A}$ is invertible, the discrete-time state transition matrix $\mathbf{\Phi}(t, t_0)$ is not invertible. An immediate consequence is that the semigroup property holds only forward in time; that is, $\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t, s)\mathbf{\Phi}(s, t_0)$ for $t \geq s \geq t_0$. Due to time-invariance, we have $\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t - t_0, 0)$.

### 6.5.2 How to Determine Powers of Matrices

Given an $\mathbf{A} \in \mathbb{R}^{n\times n}$, if $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a Jordan canonical form of $\mathbf{A}$, then we have

$$\mathbf{A}^t = (\mathbf{P}\mathbf{J}\mathbf{P}^{-1})^t = \mathbf{P}\mathbf{J}^t\mathbf{P}^{-1}.$$

In particular, if $\mathbf{J} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, then $\mathbf{\Phi}(t, 0) = \mathbf{A}^t = \mathbf{P}\operatorname{diag}\{\lambda_1^t, \ldots, \lambda_n^t\}\mathbf{P}^{-1}$. Also, by the Cayley-Hamilton Theorem there exist $\beta_i(\cdot)$, $i = 1, \ldots, n-1$, such that

$$\mathbf{A}^t = \sum_{i=0}^{n-1} \beta_i(t)\mathbf{A}^i, \quad t = 0,\ 1,\ \ldots. \tag{6.12}$$

Thus, as in the continuous-time case, one can determine $\mathbf{A}^t$ by letting

$$f(s) = s^t \quad \text{and} \quad g(s) = \beta_0(t) + \beta_1(t)s + \cdots + \beta_{n-1}(t)s^{n-1},$$

and then solving (6.9) for the terms $\beta_i(t)$. Finally, one can use the fact that $\mathbf{\Phi}(\cdot, 0)$ is the inverse $z$-transform of $z(z\mathbf{I}-\mathbf{A})^{-1} = (\mathbf{I}-z^{-1}\mathbf{A})^{-1}$ to determine $\mathbf{A}^t$. If $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of $\mathbf{A}$, with corresponding algebraic multiplicities $m_1, \ldots, m_p$ such that $\sum_{i=1}^p m_i = n$, we have

$$z(z\mathbf{I} - \mathbf{A})^{-1} = z\sum_{j=1}^{p}\sum_{k=0}^{m_j-1} \frac{k!}{(z-\lambda_j)^{k+1}}\mathbf{A}_{jk}$$

and hence

$$\mathbf{A}^t = \sum_{j=1}^{p}\sum_{k=0}^{\min\{m_j-1, t\}} \frac{t!}{(t-k)!}\lambda_j^{t-k}\mathbf{A}_{jk},$$

where $\mathbf{A}_{jk} \in \mathbb{C}^{n\times n}$ are the matrices of partial fraction expansion coefficients. If $\mathbf{A}_{jk}$ is nonzero, then $\frac{t!}{(t-k)!}\lambda_j^{t-k}$ is called a *mode* of the system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$. We have $\mathbf{x}(t) \to 0$ under arbitrary initial conditions if each mode converges to zero (i.e., each $\lambda_j$ has a magnitude less than 1).

## References

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