THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 5
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Due date of the homework is: Friday, October 13th @ 11:59pm.

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1. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

(a) Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.

$$x(2) = A^2 x(0)$$

$$x(0) = (A^2)^{-1} x(2)$$

$$x(0) = \left(\begin{bmatrix} 2a & 1 \\ 0 & a^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(0) = \frac{1}{2a^3} \begin{bmatrix} a^2 & -1 \\ 0 & 2a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} \frac{1}{2a} & \frac{-1}{2a^3} \\ 0 & \frac{1}{a^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} \frac{a^2-1}{2a^3} \\ \frac{1}{2a^2} \end{bmatrix}$$

(b) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k} 1^+(k)$ and $x(0) = 0$.

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

$$x(n) = \sum_{j=0}^{n-1} \begin{bmatrix} ja^{j-1} & 1 \\ 0 & a^n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a^{-(n-1-j)}$$

$$x(n) = (a^{1-n}) \begin{bmatrix} \frac{1}{2} \sum_{j=0}^{n-1} 2ja^{2j-1} \\ 0 \end{bmatrix}$$

$$\frac{1}{2} \sum_{j=0}^{n-1} 2ja^{2j-1} = (\frac{1}{2}) \frac{d}{da} \left[\frac{1-(a^2)^n}{1-(a^2)} \right] = (\frac{1}{2}) \frac{1 - n(a^2)^{n-1} + (n-1)(a^2)^n}{1-(a^2)^2}$$

$$x(n) = \begin{bmatrix} (a^{1-n}) \frac{1 - na^{2n-2} + (n-1)(a^{2n})}{2-2a^4} \\ 0 \end{bmatrix}$$

2. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{D} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

(a) Find a general expression for D^k.
Since D has the form of a Jordan block, then:

$$D^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix}$$

(b) Find A^k.

$$T_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$T_2 * T_1 = I_2$$

$$A^k = (T_1 D T_2)_1 (T_1 D T_2)_2 ... (T_1 D T_2)_k$$

$$A^k = T_1 D^k T_2$$

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$A^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} + \lambda_1^k \\ \lambda_1^k & k\lambda_1^{k-1} - \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$A^k = \begin{bmatrix} \lambda_1^k + (0.5)k\lambda_1^{k-1} & -(0.5)\lambda_1^{k-1} \\ (0.5)\lambda_1^{k-1} & \lambda_1^k - (0.5)k\lambda_1^{k-1} \end{bmatrix}$$

(c) Compute $x(k)$ if the control input is null.

$$x(k) = A^k x(0)$$

$$x(k) = \begin{bmatrix} \lambda_1^k + (0.5)k\lambda_1^{k-1} & -(0.5)\lambda_1^{k-1} \\ (0.5)\lambda_1^{k-1} & \lambda_1^k - (0.5)k\lambda_1^{k-1} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$x(k) = \begin{bmatrix} k\lambda_1^{k-1} + \lambda_1^{k-1} + \lambda_1^k \\ k\lambda_1^{k-1} + \lambda_1^{k-1} - \lambda_1^k \end{bmatrix}$$

(d) Compute $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k 1^+(k)$ and $\lambda_1 = 4$.

$$x(k) = \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

$$x(k) = \sum_{j=0}^{k-1} \begin{bmatrix} 2^{2j} + j2^{2j-2-1} & -2^{2j-2-1} \\ 2^{2j-2-1} & 2^{2j} - j2^{2j-2-1} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} 2^{(k-1-j)}$$

$$x(k) = \sum_{j=0}^{k-1} \begin{bmatrix} 2^{2j+1} + j2^{2j-2} - 2^{2j-2} \\ 2^{2j-2} + 2^{2j+1} - j2^{2j-2} \end{bmatrix} 2^{k-1-j}$$

$$x(k) = \sum_{j=0}^{k-1} \begin{bmatrix} (2^k)(2^j) + (2^{k-2})(j2^{j-1}) - (2^{k-3})(2^j) \\ (2^{k-3})(2^j) + (2^k)(2^j) - (2^{k-2})(j2^{j-1}) \end{bmatrix}$$

$$\sum_{j=0}^{k-1} 2^j = \frac{1-2^k}{1-2} = 2^k - 1$$

$$\sum_{j=0}^{k-1} j2^{j-1} = \frac{1 - k2^{k-1} + (k-1)2^k}{(1-2)^2} = 1 - k2^{k-1} + (k-1)2^k = 1 - k2^{k-1} + k2^k - 2^k$$

$$x(k) = \begin{bmatrix} (2^k)(2^k - 1) + (2^{k-2})(1 - k2^{k-1} + k2^k - 2^k) - (2^{k-3})(2^k - 1) \\ (2^{k-3})(2^k - 1) + (2^k)(2^k - 1) - (2^{k-2})(1 - k2^{k-1} + k2^k - 2^k) \end{bmatrix}$$

3. This problem requires you to think deeply about the problem and to remember the linear algebra background we discussed in Module 2. Consider the following system with two inputs $\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = u(k)$ and the following dynamics:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u(k), x(0) = 0.$$

(a) By setting $u_2(k) = 0 \ \forall k$, and using $u_1(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_1(k)$ that would achieve that for $k = 0, 1, 2$.

Yes, that is possible because:

$$x(3) = \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

$$x(3) = \sum_{j=0}^{k-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^j \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(k-1-j) \\ 0 \end{bmatrix}$$

$$x(3) = \sum_{j=0}^{k-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^j \begin{bmatrix} u_1(k-1-j) \\ u_1(k-1-j) \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(2) \\ u_1(2) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(1) \\ u_1(1) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1(2) + 2u_1(1) + 3u_1(0) \\ u_1(2) + u_1(1) + u_1(0) \end{bmatrix}$$

From here, the corresponding values for $u_1(0), u_1(1), u_1(2)$ can be obtained:

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 1 & 1 & 1 & | & -1 \end{bmatrix}$$

The matrix can be processed by performing the following operations: $r_1(-1) + r_2 = r_2$, $r_2(-1) = r_2$, $r_2(-2) + r1 = r1$:

$$\begin{bmatrix} 1 & 0 & -1 & | & -3 \\ 0 & 1 & 2 & | & 2 \end{bmatrix}$$

$$u_1(2) = -3 + u_1(0)$$
$$u_1(1) = 2 - 2u_1(0)$$
$$u_1(0) \in \mathbb{R}$$

(b) By setting $u_1(k) = 0\ \forall k$, and using $u_2(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_2(k)$ that would achieve that for $k = 0, 1, 2$.

No, because:

$$Bu(k-1-j) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u_2(k-1-j) \end{bmatrix} = \begin{bmatrix} u_2(k-1-j) \\ 0 \end{bmatrix}$$

Because of the form of $A^j\ \forall j$, $x_2(3)$ will always be zero.

(c) Assume at $k = 0, 1$, only u_1 can be used and at $k = 2$, only u_2 can be used. Find the input $u(k) \forall k$ such that the state can be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$x(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_2(2) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(1) \\ u_1(1) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(0) \end{bmatrix}$$

Since a $u(k) \forall k$ is required, then $u_1(0) = u_1(1)$ and the following equations are obtained:

$$u_2(2) + 2u_1(0) + 3u_1(0) = 1$$
$$0 + u_1(0) + u_1(0) = -1$$

$$u_1(0) = u_1(1) = -\frac{1}{2}$$
$$u_2(2) = \frac{7}{2}$$

Thus, the function $u(k)$ will be as follows:

$$u(k) = \begin{bmatrix} -\frac{1}{2} 1^+(k) \\ \frac{7}{2} 1^+(k) \end{bmatrix}$$

4. You are given this system:

$$x(k+1) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), a \neq 0, b \neq 0.$$

(a) Prove that $A^k = \begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix}$.

By mathematical induction:

$$A^2 = AA = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a*a & a+a \\ 0*0 & a*a \end{bmatrix} = \begin{bmatrix} a^2 & 2a^1 \\ 0 & a^2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a*a^2 & a^2+2a^2 \\ 1*0 & a*a^2 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

(b) If $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u(k) = 0$, find $x(0)$.

$$x(2) = A^2 x(0)$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} x(0)$$

$$(A^2)^{-1} = \frac{1}{a^4} \begin{bmatrix} a^2 & -2a \\ 0 & a^2 \end{bmatrix}$$

$$\frac{1}{a^4} \begin{bmatrix} a^2 & -2a \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x(0)$$

$$\begin{bmatrix} \frac{1}{a^2} - \frac{2}{a^3} \\ \frac{1}{a^2} \end{bmatrix} = x(0)$$

(c) Find $x(k)$ if $u(k) = a^k$ and $x(0) = 0$.

$$x(k) = \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

$$x(k) = \sum_{j=0}^{k-1} \begin{bmatrix} a^j & ja^{j-1} \\ 0 & a^j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a^{k-1-j}$$

$$x(k) = \sum_{j=0}^{k-1} \begin{bmatrix} a^{k-1} \\ 0 \end{bmatrix}$$

$$x(k) = \begin{bmatrix} ka^{k-1} \\ 0 \end{bmatrix}$$

5. You're given the following DT LTV system:

$$x(k+1) = A(k)x(k) + B(k)u(k).$$

(a) Derive a system of equations whose solution gives the two inputs $u(0), u(1)$ that would drive the system from state $x(0)$ to $x(2)$.

$$x(2) = A(1)A(0)x(0) + A(1)B(0)u(0) + B(1)u(1)$$

$$(B(1))^{-1}(x(2) - A(1)A(0)x(0) - A(1)B(0)u(0)) = u(1)$$

$$(A(1)B(0))^{-1}(x(2) - A(1)A(0)x(0) - B(1)u(1)) = u(0)$$

(b) Now assume that

$$A(k) = \begin{bmatrix} 0 & 2-k \\ 0 & 0 \end{bmatrix}, B(k) = \begin{bmatrix} 2-k & 0 \\ 0 & 2-k \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the input sequence $u(0), u(1)$ that would steer the system from $x(0)$ to $x(2)$.

$$A(0) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} u(0) \right) = u(1)$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 - 2u_1(0) \\ 2 \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ \frac{-1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix} \right) = u(0)$$

$$\begin{bmatrix} 0 & 0 \\ \frac{-1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 - u_1(1) \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{-1}{2} + \frac{u_1(1)}{2} \end{bmatrix} = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

Where $u_2(0)$ and $u_1(1)$ are dependant of each other, more information is needed to derive the exact needed values, given that $x(1)$ can have any value.

6. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t)-1)\\ \dot{x}_2(t) &= x_2^2(t)+x_1(t)-3\end{aligned}$$

(a) Find all the equilibrium points of the nonlinear system.

$$0 = x_2(t)(x_1^2(t)-1)$$

$$x_{2eq} = 0$$
$$x_{1eq} = 1, -1$$

Using $x_{2eq} = 0$:

$$0 = x_1(t) - 3 \quad \Rightarrow \quad x_{eq} = \begin{bmatrix}3\\0\end{bmatrix}$$

Using $x_{1eq} = 1, -1$:

$$x_2^2 = 3+1 \quad \Rightarrow \quad x_{eq} = \begin{bmatrix}-1\\+-2\end{bmatrix}$$

$$x_2^2 = 3-1 \quad \Rightarrow \quad x_{eq} = \begin{bmatrix}1\\+-\sqrt{2}\end{bmatrix}$$

(b) Determine the stability of the system around each equilibrium point, if possible.

$$\frac{\partial}{\partial x}\dot{x}(t) = \begin{bmatrix}2x_2(t)x_1(t) & x_1^2(t)-1\\ 1 & 2x_2(t)\end{bmatrix}$$

For $x_{eq} = \begin{bmatrix}3 & 0\end{bmatrix}^T$ the system is unstable, since the eigenvalues of the matrix are equal to $+-\sqrt{8}$:

$$\begin{bmatrix}2(0)(3) & (3)^2-1\\ 1 & 2(0)\end{bmatrix} = \begin{bmatrix}0 & 8\\ 1 & 0\end{bmatrix}$$

For $x_{eq} = \begin{bmatrix}-1 & 2\end{bmatrix}^T$ the system is unstable, since the eigenvalues of the matrix are equal to $4, -4$:

$$\begin{bmatrix}2(2)(-1) & (-1)^2-1\\ 1 & 2(2)\end{bmatrix} = \begin{bmatrix}-4 & 0\\ 1 & 4\end{bmatrix}$$

the system is unstable, since the eigenvalues of the matrix are equal to $4, -4$:
For $x_{eq} = \begin{bmatrix}-1 & -2\end{bmatrix}^T$:

$$\begin{bmatrix}2(-2)(-1) & (-1)^2-1\\ 1 & 2(-2)\end{bmatrix} = \begin{bmatrix}4 & 0\\ 1 & -4\end{bmatrix}$$

For $x_{eq} = \begin{bmatrix}1 & +\sqrt{2}\end{bmatrix}^T$, the system is unstable since both eigenvalues are equal to $+2\sqrt{2}$:

$$\begin{bmatrix}2(\sqrt{2})(1) & (1)^2-1\\ 1 & 2(\sqrt{2})\end{bmatrix} = \begin{bmatrix}2(\sqrt{2}) & 0\\ 1 & 2(\sqrt{2})\end{bmatrix}$$

For $x_{eq} = \begin{bmatrix}1 & -\sqrt{2}\end{bmatrix}^T$, the system is stable since both eigenvalues are equal to $-2\sqrt{2}$:

$$\begin{bmatrix}2(-\sqrt{2})(1) & (1)^2-1\\ 1 & 2(-\sqrt{2})\end{bmatrix} = \begin{bmatrix}2(-\sqrt{2}) & 0\\ 1 & 2(-\sqrt{2})\end{bmatrix}$$

(c) Solve the same problem if the system is in discrete time:

$$\begin{aligned} x_1(k+1) &= x_2(k)(x_1^2(k)-1) \\ x_2(k+1) &= x_2^2(k)+x_1(k)-3. \end{aligned}$$

To obtain the equilibrium points of this system, Matlab's fsolve function will be used:

$$0 = x_2(k)x_1^2(k) - x_2(k) - x_1(k)$$
$$0 = x_2^2(k) + x_1(k) - 3 - x_2(k)$$

A script named "root2d.m" will be created, and the previously mentioned functions will be represented there:

```
function F = root2d(x)
F(1) = x(2)*(x(1))^2-x(2)-x(1);
F(2) = (x(2))^2+x(1)-3-x(2);
```

From here, the following commands will be executed:

```
>> fun = @root2d;
>> x0 = [1, 1];
>> x = fsolve(fun,x0)
```

For initialization $x_1 = 1, x_2 = 1$ the points of equilibrium are:

```
x =

1.2977    1.8973
```

For initialization $x_1 = 100, x_2 = 100$ the points of equilibrium are:

```
x =

0.6471   -1.1133
```

For initialization $x_1 = 1000, x_2 = 1000$ the points of equilibrium are:

```
x =

3.2254    0.3430
```

For initialization $x_1 = -1, x_2 = -1$ the points of equilibrium are:

```
x =

-1.3492   -1.6446
```

To confirm that these results are valid, the following operations will be made to use Matlab's function root:

$$x_2(x_1^2 - 1) = x_1$$
$$x_2 = \frac{x_1}{(x_1^2 - 1)}$$

$$\left(\frac{x_1}{(x_1^2-1)}\right)^2 + x_1 - 3 + \frac{x_1}{(x_1^2-1)} = 0$$

$$x_1^2 + x_1(x_1^2-1) - 3(x_1^2-1)^2 - x_1(x_1^2-1) = 0$$

$$x_1^5 - 3x_1^4 - 3x_1^3 + 7x_1^2 + 2x_1 - 3 = 0$$

```
>> x1 = roots([1 -3 -3 7 2 -3])

x1 =

3.2254
-1.3492
-0.8209
1.2977
0.6471

>> func = @(x1) (x1)/((x1^2)-1);
>> x2 = arrayfun(func,x1)

x2 =

0.3430
-1.6446
2.5177
1.8973
-1.1133
```

To determine the stability of the system, A is obtained as follows:

$$\frac{\partial}{\partial x} x(k+1) = \begin{bmatrix} 2x_2(k)x_1(k) & x_1^2(k) - 1 \\ 1 & 2x_2(k) \end{bmatrix} = A$$

For $x(k) = \begin{bmatrix} 1.2977 & 1.8973 \end{bmatrix}^T$, the eigenvalues are 5.3609 and 3.3579, thus the system is unstable:

$$\begin{bmatrix} 2(1.8973)(1.2977) & (1.2977)^2 - 1 \\ 1 & 2(1.8973) \end{bmatrix} = \begin{bmatrix} 4.9242 & 0.6840 \\ 1 & 3.7946 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} 0.6471 & -1.1133 \end{bmatrix}^T$, the eigenvalues are -1.0767 and -2.5907, for that, the system is unstable:

$$\begin{bmatrix} 2(-1.1133)(0.6471) & (0.6471)^2 - 1 \\ 1 & 2(-1.1133) \end{bmatrix} = \begin{bmatrix} -1.4408 & 0.4187 \\ 1 & -2.2266 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} 3.2254 & 0.3430 \end{bmatrix}^T$, the eigenvalues are 4.7638 and -1.8652, which means that the system is unstable:

$$\begin{bmatrix} 2(0.3430)(3.2254) & (3.2254)^2 - 1 \\ 1 & 2(0.3430) \end{bmatrix} = \begin{bmatrix} 2.2126 & 10.4032 \\ 1 & 0.6860 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} -1.3492 & -1.6446 \end{bmatrix}^T$, the eigenvalues are 4.0536 and -2.9051, which means that the system is unstable:

$$\begin{bmatrix} 2(-1.6446)(-1.3492) & (-1.3492)^2 - 1 \\ 1 & 2(-1.6446) \end{bmatrix} = \begin{bmatrix} 4.4377 & -2.8203 \\ 1 & -3.2892 \end{bmatrix}$$

For the last pair, obtained by using roots $x(k) = \begin{bmatrix} -0.8209 & 2.5177 \end{bmatrix}^T$, the eigenvalues are -3.9472 and 4.8491, meaning that the system is unstable:

$$\begin{bmatrix} 2(2.5177)(-0.8209) & (-0.8209)^2 - 1 \\ 1 & 2(2.5177) \end{bmatrix} = \begin{bmatrix} -4.1335 & -1.6738 \\ 1 & 5.0354 \end{bmatrix}$$

All the eigenvalues were obtained using Matlab, none of the equilibrium points are stable in discrete time because all the eigenvalues pair are greater than 1 or less than -1.