MCE693/793: Analysis and Control of Nonlinear Systems

Introduction to Nonlinear Controllability and Observability

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Definition of Controllability

Definition: The system

$$\dot{x} = f(x, u)$$

is (fully) controllable if given initial and final points $x(t_0)$ and x_f, we can always find an admissible control input $u(t)$ and a *finite* time t_f such that

$$\Phi(x(t_0), t_f) = x_f$$

where Φ is the flow of the differential equation

$$\dot{x} = f(x, u(t))$$

In plain words, there has to be a control that takes the system to the initial point to the final point in some finite time. Note that there is no steady-state requirement for the final point.

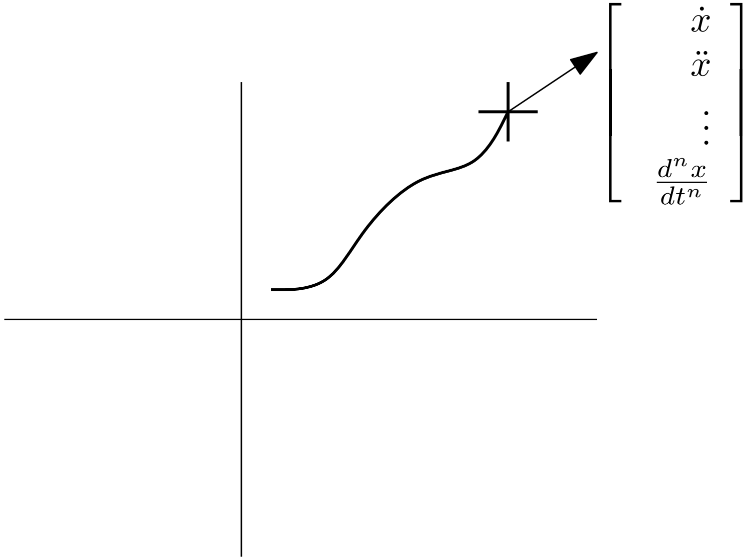
Review: Linear Controllability

Consider the LTIS

$$\dot{x} = Ax + Bu$$

with m inputs and n states.

Intuitively, the ability to drive the system state from one point to another infinitesimally close to it is related to the possibility of using a constant control input to target desired values for $\dot{x}, \ddot{x} ... \frac{dx^n}{dt^n}$ all at once.



Linear Controllability

Setting up equations:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
\ddot{x} &= A(Ax + Bu) \\
&\vdots \\
\frac{dx^n}{dt^n} &= A(A...(Ax + Bu))
\end{aligned}
$$

This can be rearranged as

$$
\left[\begin{array}{l} \dot{x} - Ax \\ \ddot{x} - A^2 x \\ \vdots \\ \frac{dx^n}{dt^n} - A^{n-1} x \end{array} \right] = \left[\begin{array}{l} B \\ AB \\ \vdots \\ A^{n-1} B \end{array} \right] u = \mathcal{C}^T u
$$

Linear Controllability...

In the above problem, x is the current state (given). $\mathcal{C}$ is the controllability matrix, and its columns

$$B_1, B_2, ..B_m, (AB)_1...(AB)_m...(A^{n-1}B)_m$$

must span n-dimensional space to find a solution for u.

In Matlab, $\mathcal{C}$ is built with » `ctrlb(A,B)`.

Linear systems theory shows that a LTIS is controllable if and only if it is so between $x_0 = 0$ and an arbitrary x_f.

Definition of Observability

Definition: The system

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{y} &= h(x) \end{aligned}$$

is observable if $y = 0$ implies $x = 0$. A weaker definition is *detectability*, where $y = 0$ implies that $x \to 0$ as $t \to \infty$.

When a system is observable, the initial state can be uniquely determined from $y(t)$ and $u(t)$ for $t \in [t_0, t_f]$, where t_f is some finite time.

Review: Linear Observability

Consider the LTI system with m inputs, p outputs and n states:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{y} &= Cx + Du\end{aligned}$$

Again, suppose a constant control is applied from x (unknown initial point) for an infinitesimal time. We want to find x by observing y and its derivatives up to the $n-1$-th order.

Linear Observability

Setting up equations ($\dot{u} = 0$):

$$\begin{aligned}
\dot{y} &= Cx + Du \\
\dot{y} &= C(Ax + Bu) \\
&\vdots \\
\frac{dy^{n-1}}{dt^{n-1}} &= C(A...(Ax + Bu)) = CA^{n-1}x + CBu
\end{aligned}$$

This can be rearranged as

$$\begin{bmatrix} y - Du \\ \dot{y} - CBu \\ \vdots \\ \frac{dy^{n-1}}{dt^{n-1}} - CBu \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = \mathcal{O}x$$

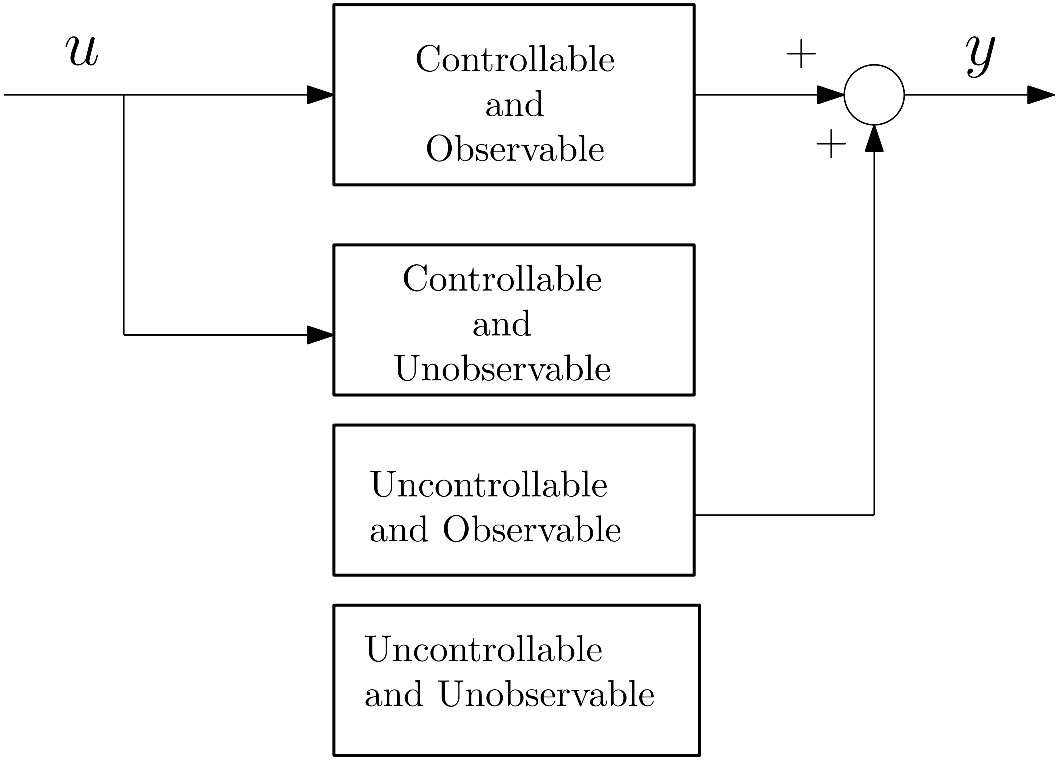
Linear Observability...

The observability matrix ′ must be full rank to find a solution for x.

In Matlab, $\mathcal{O}$ is built with » `obsvb(A,B)`.

The Kalman Decomposition

For linear systems, we can always find a linear transformation that reveals the observable/controllable subspaces in the system.



Kalman Controllability Decomposition

Given

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{y} &= Cx\end{aligned}$$

there is a transformation $z = Tx$ with T orthogonal $(T^{-1} = T')$ such that the transformed system has the form

$$\begin{aligned}\dot{x} &= \left[\begin{array}{c|c} A_{nc} & 0 \\ \hline A_{21} & A_c \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hline B_c \end{array}\right] u \\ y &= [C_{nc} | C_c] x\end{aligned}$$

In Matlab, use `ctrbf(A,B,C)`.

When A_{nc} is Hurwtiz, the system is *stabilizable*.

Kalman Observability Decomposition

Given

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{y} &= Cx\end{aligned}$$

there is a transformation $z = Tx$ with T orthogonal $(T^{-1} = T')$ such that the transformed system has the form

$$\begin{aligned}\dot{x} &= \left[\begin{array}{c|c} A_{no} & A_{12} \\ \hline 0 & A_o \end{array}\right] x + \left[\begin{array}{c} B_{no} \\ \hline B_o \end{array}\right] u \\ y &= [0 | C_o] x\end{aligned}$$

In Matlab, use `obsvf(A,B,C)`.

When A_{no} is Hurwtiz, the system is *detectable*.

Lie derivative of a scalar function with respect to a vector field

Let $\mathcal{M}$ be a subset of $\mathbb{R}^n$. Let $f : \mathcal{M} \mapsto \mathbb{R}^n$ be a smooth vector field and let $h : \mathcal{M} \mapsto \mathbb{R}$ a smooth scalar function. The *Lie derivative* of h with respect to f, denoted $L_f h$ is the directional derivative in the direction of f:

$$L_f h = \nabla h f$$

If $\dot{x} = f(x)$ is a nonlinear system and $V(x)$ is a Lyapunov function candidate, $\dot{V} = L_f V$.

Lie derivatives as operators

The Lie derivative can be applied recursively:

$$L_f(L_f h) = \nabla(L_f h) f = L_f^2 h$$

Also, we can use various vector fields:

$$L_g(L_f h) = \nabla(L_f h) g = L_g L_f h$$

The Lie derivative is not commutative:

For $h(x, y) = x - y^2$, $g(x, y) = [x\ y]^T$ and $f(x, y) = [y\ x^2]^T$, calculate $L_g L_f h$ and $L_f L_g h$

The Lie Bracket

Let f and g be two smooth vector fields. The Lie bracket of f and g is another vector field defined by

$$[f, g] = \nabla g f - \nabla f g$$

The notation $\mathsf{ad}_f g$ is also used for $[f, g]$ (given a fixed f, $\mathsf{ad}_f g$ is the adjoint action of f on the set of all smooth vector fields on $\mathcal{M}$.

The Lie bracket defines a non-associative *algebra* of smooth vector fields on a manifold). The algebraic properties are:

1. Bilinearity: $[\alpha_1 f_1 + \alpha_2 f_2, g] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g]$
2. Antisymmetry: $[f, g] = -[g, f]$
3. Jacobi identity: $L_{[f,g]} h = L_{\mathsf{ad}_f g} h = L_f L_g h - L_g L_f h$

Combining the given form of bilinearity with antisymmetry shows that

$$[f, \alpha_1 g_1 + \alpha_2 g_2] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g]$$

Recursive Lie Brackets

$$\mathsf{ad}_{f^2} g = [f, \mathsf{ad}_f g]$$

This can be worked out using the Jacobi identity:

$$\mathsf{ad}_{f^2 g} h = L_{f^2} L_g h - 2 L_f L_g L_f h + L_g L_{f^2} h$$

Exercise: Follow the proof of the Jacobi identity in Slotine and Li and use it to verify the above formula.

Back to Linear Controllability

Consider

$$\dot{x} = Ax + Bu = Ax + B_1u_1 + B_2u_2 + ...B_mu_m$$

We revisit the problem of "targeting" desired values for $\dot{x}, \ddot{x}, ... \frac{d^nx}{dt^n}$ simultaenously, using a constant control.

We show that

$$\mathcal{C} = [B_1, ...B_m, \mathsf{ad}_f B_1, ...\mathsf{ad}_f, B_m, ...\mathsf{ad}_{f^{n-1}} B_m]$$

Again, $\mathcal{C}$ must have n linearly independent columns.

Nonlinear Controllability

Consider the class of affine control systems

$$\dot{x} = f(x) + g(x)u$$

where the columns g_i of g span $\mathbb{R}^m$.

Hunt's theorem (1982): The nonlinear system is (locally) controllable if there exists an index k such that

$$\mathcal{C} = [g_1, ...g_m, \mathsf{ad}_f g_1, ...\mathsf{ad}_f g_m, ...\mathsf{ad}_{f^k} g_1, ...\mathsf{ad}_{f^k} g_m]$$

has n linearly independent columns.

Hunt, L.R., "Sufficient Conditions for Controllability", *IEEE Trans. Circuit and Systems*, May 1982.

Example (Hunt)

$$
\begin{aligned}
\dot{x}_1 &= \cos(\theta)x_3 + \sin(\theta)x_4 \\
\dot{x}_2 &= -\sin(\theta)x_3 + \cos(\theta)x_4 \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= u_2
\end{aligned}
$$

with $\theta = \sqrt{x_1^2 + x_2^2}$.
Notes:

1. Controllability is only local. It can be verified near a point, lost away from that point.
2. **Linearization does not preserve local controllability properties!**

Back to Linear Observability

With $y_i = C_i x = h_i(x)$, we can take successive derivatives of y using the Lie derivative, using vector field $f = Ax$:

$$\frac{d^k y_i}{dt^k} = L_f^k h_i$$

Define

$$G = \begin{bmatrix} L_f^0 h_1 & \dots L_f^0 h_p \\ \vdots & \vdots \\ L_f^{n-1} h_1 & \dots L_f^{n-1} h_p \end{bmatrix}$$

In the linear case, G is:

$$G = \begin{bmatrix} C_1 x & \dots C_p x \\ \vdots & \vdots \\ C_1 A^{n-1} x & \dots C_p A^{n-1} x \end{bmatrix}$$

where C_i are the rows of C, $i = 1, 2 ... p$.

Linear Observability...

Form a matrix dG with the gradients of the Lie derivatives of G:

$$dG = \begin{bmatrix} dL_f^0 h_1 & \dots dL_f^0 h_p \\ \vdots & \vdots \\ dL_f^{n-1} h_1 & \dots dL_f^{n-1} h_p \end{bmatrix}$$

In the linear case, dG becomes the observability matrix.

Nonlinear Weak Observability

Theorem (Hermann and Krener, 1977): Let

$$\begin{aligned}\dot{x} &= f(x,u)\\ \dot{y} &= h(x)\end{aligned}$$

Let G be the set of all finite linear combinations formed with the Lie derivatives of h_1, h_2,...h_p with respect to f and constant u. Let dG denote the set of the gradients of the elements of G.

The system is weakly (locally) observable if dG contains n linearly independent vectors.

Hermann, R. and Krener, A.J., Nonlinear Controllability and Observability, *IEEE Trans. Automatic Control*, AC-22, n5, 1977.

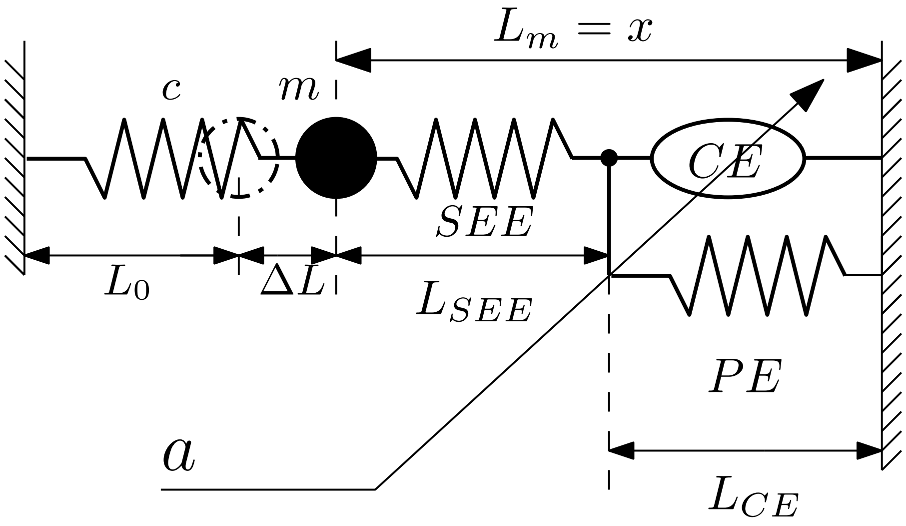
Example

$$
\begin{aligned}
\dot{x}_1 &= \frac{x_1^2}{2} + e^{x_2} + x_2 \\
\dot{x}_2 &= x_1^2 \\
y &= x_1
\end{aligned}
$$

Note that x_2 can be found from y and $\dot{y}$, and x_1 can obviously be found from y. Therefore we now that the system is observable. We use the above technique to show weak observability.

Example: Muscle-Driven System

Consider a mass-spring system driven by a Hill muscle model:



where $L_{CEE} + L_{SEE} = L_m = x_1$. Constant β is defined by

$$\beta = x_{eq} + \Delta L$$

where x_{eq} is the equilibrium muscle length and ΔL is the corresponding elongation of the restraining spring of constant c.

Muscle-Driven System...

The dynamics of the system are given by

$$\begin{aligned} \dot{x}_1 &= x_2 & (1) \\ \dot{x}_2 &= \frac{1}{m}\left[-\Phi_S(L_{SEE}) + c(\beta - x_1)\right] & (2) \\ \dot{L}_{SEE} &= x_2 + u & (3) \end{aligned}$$

where u is the contraction speed of the CE, regarded as control input in this simplified example.

$\Phi_S(L_{SEE})$ is the force-length relationship for the series elasticity, which contains a deadzone. We use the above results to verify local controllability and weak observability with $y = x_1$.

u is related to the muscle activation a by an algebraic equation. Controllability analysis is meaningful with u as the input.