Pole placement via state feedback

$$\dot{x} = Ax + Bu, \quad x \in \Re^n, \; u \in \Re$$
$$y = Cx + Du$$

- Poles of transfer function are eigenvalues of A
- Pole locations affect system response
  - stability
  - convergence rate
  - command following
  - disturbance rejection
  - noise immunity ....
- Assume $x(t)$ is available
- Design $u = -Kx + v$ to affect closed loop eigenvalue:

$$\dot{x} = Ax + B(-Kx + v) = \underbrace{(A - BK)}_{Ac} x + Bv$$

  such that eigenvalues of A_c are $\sigma_1, \ldots, \sigma_n$.

- $K =$ state feedback gain; $v =$ auxiliary input.

Controller Canonical Form (SISO)

A system is said to be in **controller (canonical) form** if:

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

What is the relationship between a_i, $i = 0, \ldots, n-1$ and eigenvalues of A?

- Consider the characteristic equation of A:

$$\psi(s) = det(sI - A) = det \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & a_2 \end{pmatrix}$$

- Eigenvalues of A, $\lambda_1, \ldots, \lambda_n$ are roots of $\psi(s) = 0$.

$$\psi(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

- Therefore, **if we can arbitrarily choose** $a_0, \ldots, a_{n-1}$**, we can choose the eigenvalues of** A.

Target characteristic polynomial

- Let desired eigenvalues be σ_1, σ_2, $\ldots \sigma_n$.
- Desired characteristic polynomial:

$$\bar{\psi}(s) = \Pi_{i=1}^{n}(s-\sigma_i) = s^n+\bar{a}_{n-1}s^{n-1}+\ldots+\bar{a}_1 s+\bar{a}_0$$

Some properties of characteristics polynomial for its proper design:

- If σ_i are in conjugate pair (i.e. for complex poles, $\alpha \pm j\beta$), then $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}$ are real numbers; and vice versa.
- Sum of eigenvalues: $\bar{a}_{n-1} = -\sum_{i=1}^{n} \sigma_i$
- Product of eigenvalues: $\bar{a}_0 = (-1)^n \Pi_{i=1}^{n}\sigma_i$
- If $\sigma_1, \ldots, \sigma_n$ all have negative real parts, then $\bar{a}_i > 0$ $i = 0, \ldots n-1$.
- If any of the polynomial coefficients is non-positive (negative or zero), then one or more of the roots have nonnegative real parts.

Consider state feedback:

$$u = -Kx + v$$
$$K = [k_0,\ k_1,\ k_2]$$

Closed loop equation:

$$\dot{x} = Ax + B(-Kx + v) = \underbrace{(A - BK)}_{Ac} x + Bv$$

with

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_0) & -(a_1 + k_1) & -(a_2 + k_2) \end{pmatrix}$$

Thus, to place poles at $\sigma_1, \ldots, \sigma_n$, choose

$$\bar{a}_0 = a_0 + k_0 \Rightarrow k_0 = \bar{a}_0 - a_0$$
$$\bar{a}_1 = a_1 + k_1 \Rightarrow k_1 = \bar{a}_1 - a_1$$
$$\vdots$$
$$\bar{a}_{n-1} = a_{n-1} + k_{n-1} \Rightarrow k_{n-1} = \bar{a}_{n-1} - a_{n-1}$$

Conversion to controller canonical form

$$\dot{x} = Ax + Bu$$

- If we can convert a system into controller canonical form via invertible transformation $T \in \Re^{n\times n}$:

$$z = T^{-1}x; \quad A_z = T^{-1}AT, \quad B_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = T^{-1}B$$

where $\dot{z} = A_z z + B_z u$ is in controller canonical form:

$$A_z = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & \dots & 0 & 1 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \quad B_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

we can design state feedback

$$u = -K_z z + v$$

to place the poles (for the transformed system)

- Since A and $A_z = T^{-1}AT$ have same characteristic polynomials:

$$
\begin{aligned}
det(\lambda I - T^{-1}AT) &= det(\lambda T^{-1}T - T^{-1}AT) \\
&= det(T)det(T^{-1})det(\lambda I - A) \\
&= det(\lambda I - A)
\end{aligned}
$$

The control law:

$$u = -K_z T^{-1}x + v = -Kx + v$$

where $K = K_z T^{-1}$ places the poles at the desired locations.

Theorem *For the single input LTI system, $\dot{x} = Ax + Bu$, there is an invertible transformation T that converts the system into controller canonical form if and only if the system is controllable.*

Proof:

- "Only if": If the system is not controllable, then using Kalman decomposition, there are modes that are not affected by control. Thus, eigenvalues associated with those modes cannot be changed. This means that we cannot tranform the system into controller canonical form, since otherwise, we can arbitrarily place the eigenvalues.

- "If": Let us construct T. Take $n = 3$ as example, and let T be:

$$T = [v_1 \mid v_2 \mid v_3]$$

$$A = T\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} T^{-1}; \qquad B = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This says that $v_3 = B$.

Note that A_z is determined completely by the characteristic equation of A.

$$AT = T\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \tag{11}$$

Now consider each column of (11) at a time, starting from the last. This says that:

$$A \cdot v_3 = v_2 - a_2 v_3, \Rightarrow \; v_2 = Av_3 + a_2 v3 = AB + a_2 B$$

Having found v_2, we can find v_1 from the 2nd column from (11). This says,

$$\begin{aligned} & A \cdot v_2 = v_1 - a_1 v_3, \\ \Rightarrow \quad & v_1 = Av_2 + a_1 v3 = A^2 B + a_2 AB + a_1 B \end{aligned}$$

- Now we check if the first column in (11) is consistent with the v_1, v_2 and v_3 we had found. It says:

$$A \cdot v_1 + a_0 v_3 = 0.$$

Is this true? The LHS is:

$$\begin{aligned} A \cdot v_1 + a_o v_3 =& A^3 B + a_2 A^2 B + a_1 AB + a_0 B \\ =& (A^3 + a_2 A^2 + a_1 A + a_0 I) B \end{aligned}$$

Since $\psi(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is the characteristic polynomial of A, by the Cayley Hamilton Theorem, $\psi(A) = 0$, so $A^3 + a_2 A^2 + a_1 A + a_0 I = 0$. Hence, $A \cdot v_1 + a_0 v_3 = 0$.

- To complete the proof, we need to show that if the system is controllable, then T is non-singular. Notice that

$$T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} \begin{pmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that T is non-singular if and only if the controllability matrix is non-singular. ∎

Summary procedure for pole placement:

- Find characteristic equation of A,

$$\psi_A(s) = det(sI - A)$$

- Define the target closed loop characteristic equation $\psi_{A_c}(s) = \Pi_{i=1}^{n}(s-\sigma_i)$, where σ_i are the desired pole locations.

- Compute v_n, v_{n-1} etc. successively to contruct T,

$$\begin{aligned} v_n &= b \\ v_{n-1} &= Av_n + a_{n-1}b \\ &\vdots \\ v_k &= Av_{k+1} + a_k b \end{aligned}$$

- Find state feedback for transformed system: $z = T^{-1}x$:

$$u = K_z z + v$$

- Transform the feedback gain back into original coordinates:

$$u = Kx + v; \qquad K = K_z T^{-1}.$$

Arbitrarily pole placement???

Consider system

$$\dot{x} = -x + u$$
$$y = x$$

Let's place pole at $s = -100$ and match the D.C. gain.

Consider $u = -Kx + 100v$. The using $K = 99$,

$$\dot{x} = -(1 + K)x + 100v = -100(x - v).$$

This gives a transfer function of

$$X(s) = \frac{100}{s + 100}V(s).$$

If $v(t)$ is a step input, and $x(0) = 0$, then $u(0) = 100$ which is very large. Most likely saturates the system.

Thus, due to physical limitations, it is not practically possible to achieve arbitrarily fast eigen values.

Pole placement - multi-input case

$$\dot{x} = Ax + Bu$$

with $B \in \Re^{n\times m}$, $m > 1$.

- The choice of eigenvalues do not uniquely specify the feedback gain K.
- Many choices of K lead to same eigenvalues but different eigenvectors.
- Possible to assign eigenvectors in addition to eigenvalues.

Hautus Keymann Lemma

Let (A, B) be controllable. Given any $b \in Range(B)$, there exists $F \in \Re^{m\times n}$ such that $(A + BF, b)$ is controllable.

Suppose that $b = B \cdot g$, where $g \in \Re^m$.

- Inner loop control:

$$u = Fx + gv \quad \Rightarrow \dot{x} = (A + BF)x + bv$$

- Outer loop SI control:

$$v = -kx + v_1$$

where k is designed for pole-placement (using technique previously given).

- It is interesting to note that generally, it may not be possible to find a $b \in \Re^n \in Range(B)$ such that (A, b) is controllable. For example: for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we cannot find a $g \in \Re^m$ such that (A, Bg) is controllable. We can see this by applying the PBH test with $\lambda = 2$. The reason for this is that A has repeated eigenvalues at $\lambda = 2$ with more than one independent eigenvector. The same situation applies, if A is semi-simple, with repeated eigenvalues.

- What the Hautus Keymann Theorem says is that it is possible after preliminary state feedback using a matrix F. In fact, generally, most F matrices will make $\bar{A} = A - BF$ has distinct eigenvalues. This makes it possible to avoid the parasitic situation mentioned above, so that one can find $g \in \Re^m$ so that $(\bar{A}, Bg)$ is controllable.

Generally, eigenvalue assignment for a multiple input system is not unique. There will be some possibilities of choosing the eigenvectors also. However, for the purpose of this class, we shall use the optimal control technique to resolve the issue of choosing appropriate feedback gain K in $u = -v + Kx$. The idea is that K will be picked based on some performance criteria, not to just to be placed exactly at some a-priori determined locations.

State feedback for time varying system

The pole placement technique is appropriate only for linear time invariant systems. How about linear time varying systems, such as obtained by linearizing a nonlinear system about a trajectory?

- Use least norm control
- Make use of *uniform controllability*

Consider the modified controllability (to zero) grammian function:

$$H_\alpha(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau) e^{-4\alpha(\tau - t_0)} d\tau.$$

Note that for $\alpha = 0$, $H_\alpha(t_0, t_1) = W_{c,[t_0,t_1]}$, the controllability (to zero) grammian.

Theorem *If $A(\cdot)$ and $B(\cdot)$ are piecewise continuous and if there exists $T > 0$, $h_M \geq h_m > 0$ s.t. for all $t \geq 0$,*

$$0 < h_m I \leq H_0(t, t+T) \leq h_M I$$

then for any $\alpha > 0$, the linear state feedback,

$$u(t) = -F(t)x(t) = -B^T(t)H_\alpha(t, t+T)^{-1}x(t)$$

will result in a closed loop system:

$$\dot{x} = (A(t) - B(t)F(t))x(t)$$

such that for all $x(t_0)$, and all t_0,

$$\|x(t)e^{\alpha t}\| \to 0$$

i.e. $x(t) \to 0$ exponentially ($x(t) \leq ke^{-\alpha t}$ for some k).

Remarks:

- H_α can be computed beforehand.
- Can be applied to periodic systems, e.g. swimming machine.

- α is used to choose the decay rate.

- The controllability to 0 map on the interval (t_0, t_1) $L_{c,[t_0,t_1]}$ is:

$$u(\cdot) \mapsto L_{c,[t_0,t_1]}[u(\cdot)] := -\int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)u(\tau)d\tau$$

The least norm solution that steers a state $x(t_0)$ to $x(t_1) = 0$ with respect to the cost function:

$$J_{[t_0,t_1]} = \int_{t_0}^{t_1} u^T(\tau)u(\tau)exp(4\alpha(\tau - t_0))d\tau$$

is:

$$u(\tau) = -e^{-4\alpha(\tau - t_0)}B^T(\tau)\Phi(t_0,\tau)^T H_\alpha(t_0,t_1)^{-1}x(t_0).$$

and when evaluated at $\tau = t_0$,

$$u(t_0) = -B^T(t_0)H_\alpha(t_0,t_1)^{-1}x(t_0).$$

Thus, the proposed control law is the least norm control evaluated at $\tau = t_0$. By relating this to a moving horizon $[t_0, t_1] = [t, t+T]$, where t

continuously increases, the proposed control law is the moving horizon version of the least norm control. This avoids the difficulty of receding horizon control where the control gain can become infinite when $t \to t_f$.

- Proof is based on a Lyapunov analysis typical of nonlinear control, and can be found in [Desoer and Callier, 1990, p. 231]

Observer Design

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The observer problem is that given $y(t)$ and $u(t)$ can we determine the state $x(t)$?

Openloop observer

Suppose that A has eigenvalues on the LHP (stable). Then an open loop obsever is a simulation:

$$\dot{\hat{x}} = A\hat{x} + Bu$$

The observer error dynamics for $e = \hat{x} - x$ are:

$$\dot{e} = Ae$$

Since A is stable, $e \to 0$ exponentially.

The problem with open loop observer is that they do not make use of the output $y(t)$, and also it will not work in the presence of disturbances or if A is unstable.

Closed loop observer by output injection

Luenberger Observer

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \tag{12}$$

This looks like the open loop observer except for the last term. Notice that $C\hat{x} - y$ is the output prediction error, also known as the innovation. L is the observer gain.

Let us analyse the error dynamics $e = \hat{x} - x$. Subtracting the observer dynamics by the plant dynamics, and using the fact that $y = Cx$,

$$\dot{e} = Ae - L(C\hat{x} - Cx) = (A - LC)e.$$

If $A - LC$ is stable (has all its eigenvalues in the LHP), then $e \to 0$ exponentially.

The estimated state dynamics are:

$$\begin{pmatrix} \hat{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A & -LC \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} \hat{x} \\ e \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

Notice that the observer error e is not controllable from u.

Design of observer gain L:

We use eigenvalue assignment technique to choose L. i.e. choose L so that the eigenvalues of $A - LC$ are at the desired location, $p_1, p_2, \ldots, p_n$.

Fact: *Let $F \in \Re^{n\times n}$. Then, $det(F) = det(F^T)$*

Therefore,

$$det(\lambda I - \bar{F}) = det(\lambda I - \bar{F}^T).$$

Hence, F and F^T have the same eigenvalues. So choosing L to assign the eignvalues of $A - LC$ is the same as choosing L to assign the eigenvalues of

$$(A - LC)^T = A^T - C^T L^T$$

We know how to do this, since this is the state feedback problem for:

$$\dot{x} = A^T x + C^T u, \qquad u = v - L^T C.$$

The condition in which the eigenvalues can be placed arbitrarily is that (A^T, C^T) is controllable. However, from the PBH test, it is clear that:

$$rank\begin{pmatrix}\lambda I - A^T & C^T\end{pmatrix} = rank\begin{pmatrix}\lambda I - A \\ C\end{pmatrix}$$

the LHS is the controllability test, and the RHS is the observability test. Thus, the observer eigenvalues can be placed arbitrarily iff (A, C) is observable.

Properties of observer

- The observer is unbiased. The transfer function from u to $\hat{x}$ is the same as the transfer function from u to x.

- The observer error $e = \hat{x} - x$ is uncontrollable from the control u. This is because,

$$\dot{e} = (A - LC)e$$

  no matter what the control u is.

- Let $\nu(t) := y(t) - C\hat{x}(t)$ be the innovation. Since $\nu(t) = Ce(t)$, the transfer function from $u(t)$ to $\nu(t)$ is 0.

- This has the significance that feedback control of the innovation of the form

$$U(s) = -K\hat{X}(s) - Q(s)\nu(s)$$

  where $A - BK$ is stable, and $Q(s)$ is *any* stable controller (i.e. $Q(s)$ itself does not have any unstable poles), is necessarily stable. In fact, any stabilizing controller is of this form! (see Goodwin et al. Section 18.6 for proof of necessity)

- From Eq.12, the transfer function of the observer from u and y to $\hat{x}$ is:

$$\hat{X}(s) = T_1(s)U(s) + T_2(s)Y(s)$$

where

$$T_1(s) := (sI - A + LC)^{-1}B$$
$$T_2(s) := (sI - A + LC)^{-1}L$$

Both $T_1(s)$ and $T_2(s)$ have the same denominator, which is the characteristic polynomial of the observer dynamics, $\Psi_{obs}(s) = E(s) = det(sI - A + LC)$.

- With $Y(s)$ given by $G_o(s)U(s)$ where $G_o(s) = C(sI - A)^{-1}B$ is the open loop plant model, the transfer function from $u(t)$ to $\hat{X}(t)$ is

$$\begin{aligned}\hat{X}(s) &= [T_1(s) + T_2(s)G_0(s)]\, U(s)\\ &= (sI - A)^{-1}BU(s)\end{aligned}$$

i.e. the same as the open loop transfer function, from $u(t)$ to $x(t)$. In this sense, the observer is unbiased.

This fact can be computed algebraically using $T_1(s)$ and $T_2(s)$. A simpler method is to compute the Laplace transform of the estimated state dynamics:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A & -LC \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} \hat{x} \\ e \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

Where should the observer poles be ?

Theoretically, the observer error will decrease faster if the eigenvalues of the $A - LC$ are further to the left (more negative). However, effects of measurement noise can be filtered out if eigenvalues are slower. A rule of thumb is that if noise bandwidth is $Brad/s$, the fastest eigenvalue should be greater than $-B$ (i.e. slower than the noise band). This way, observer acts as a filter.

If observer states are used for state feedback, then the slowest eigenvalues of $A - LC$ should be faster than the eigenvalue of the state feedback system $A - BK$.

Observer state feedback

Feedback is implemented by assuming that the observer state estimate is the actual state:

$$u = v - K\hat{x}$$

where v is an exogenous control. The closed loop system is given by:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - BK - LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} v$$

Applying the coordinate transform

$$e = \hat{x} - x$$

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & -BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} v$$

Remarks:

- **Seperation principle** - the set of eigenvalues of the complete system is the union of the eigenvalues of the state feedback system and the eigenvalues of the observer system. Hence, state feedback and observer can in prinicple be designed separately.

$$\{\text{eigen values}\} = eig(A - BK) \cup eig(A - LC)$$

- Using observer state feedback, the transfer function from v to x is the same as in state feedback system:

$$X(s) = (sI - (A - BK))^{-1} BV(s) + \frac{f(s,0)}{E(s)}$$

where $E(s) = det(sI - A + LC)$. Note that the second term corresponds to a signal that vanishes as long as $A - LC$ is stable.

- The controller itself satisfies:

$$\frac{L(s)}{E(s)} U(s) = -\frac{P(s)}{E(s)} Y(s) + V(s) \tag{13}$$

where

$$\frac{L(s)}{E(s)} = 1 + KT_1(s) = \frac{det(sI - A + LC + BK)}{E(s)}$$

$$\frac{P(s)}{E(s)} = KT_2(s) = \frac{KAdj(sI - A)J}{E(s)}$$

$$\frac{P(s)}{L(s)} = K(sI - A + LC + BK)^{-1}L$$

[See Goodwin pp. 512 for proof]

- Controller can be written as a two degree of freedom controller form:

$$U(s) = \frac{E(s)}{L(s)}\left(V(s) - \frac{P(s)}{E(s)}Y(s)\right)$$

- Or as a 1 degree of freedom controller form:

$$U(s) = \frac{P(s)}{E(s)}\left(R(s) - Y(s)\right)$$

where $V(s) = \frac{P(s)}{E(s)}R(s)$.

Innovation feedback

The innovation is the output prediction error:

$$\nu := y - C\hat{x} = -Ce$$

Therefore,

$$\begin{aligned}\nu(s) &= Y(s) - C\hat{X}(s) \\ &= Y(s) - CT_1(s)U(s) - CT_2(s)Y(s) \\ &= (1 - CT_2(s))Y(s) - CT_1(s)U(s)\end{aligned}$$

where

$$\begin{aligned}T_1(s) &= (sI - A + LC)^{-1}B \\ T_2(s) &= (sI - A + LC)^{-1}L\end{aligned}$$

In transfer function form:

$$\frac{L(s)}{E(s)}U(s) = -\frac{P(s)}{E(s)}Y(s)$$

$$
\begin{aligned}
G(s) &= \frac{B_o(s)}{A_o(s)} = \frac{C Adj(sI - A)B}{det(sI - A)} \\
E(s) &= det(sI - A + LC) \\
F(s) &= det(sI - A + BK) \\
L(s) &= det(sI - A + LC + BK) \\
P(s) &= K Adj(sI - A)L \\
\frac{P(s)}{L(s)} &= K\left[sI - A + LC + BK\right]^{-1} L
\end{aligned}
$$

Then, it can be shown (see Goodwin P545) that the innovation

$$\nu(s) = \frac{A_o(s)}{E(s)}Y(s) - \frac{B_o(s)}{E(s)}U(s)$$

Consider now that observer state feedback augmented with innovation feedback,

$$u = v - K\hat{x} - Q_u(s)\nu$$

where $Q_u(s)\nu$ is ν filtered by the stable filter $Q_u(s)$

(to be designed). Then,

$$\frac{L(s)}{E(s)}U(s) = -\frac{P(s)}{E(s)}Y(s) - Q_u(s)\left[\frac{A_o(s)}{E(s)}Y(s) - \frac{B_o(s)}{E(s)}U(s)\right]$$

The nominal sensitivity functions, which define the robustness and performance criteria, are modified affinely by $Q_u(s)$:

$$S_o(s) = \frac{A_o(s)L(s)}{E(s)F(s)} - Q_u(s)\frac{B_o(s)A_o(s)}{E(s)F(s)} \tag{14}$$

$$T_o(s) = \frac{B_o(s)P(s)}{E(s)F(s)} + Q_u(s)\frac{B_o(s)A_o(s)}{E(s)F(s)} \tag{15}$$

For plants that are open-loop stable with tolerable pole locations, we can set $K = 0$ so that

$$F(s) = A_o(s)$$
$$L(s) = E(s)$$
$$P(s) = 0$$

so that

$$S_o(s) = 1 - Q_u(s)\frac{B_o(s)}{E(s)}$$

$$T_o(s) = Q_u(s)\frac{B_o(s)}{E(s)}$$

In this case, it is common to use $Q(s) := Q_u(s)\frac{A_o(s)}{E(s)}$ to get the formulae:

$$S_o(s) = 1 - Q(s)G_o(s) \tag{16}$$

$$T_o(s) = Q(s)G_o(s) \tag{17}$$

Thus the design of $Q_u(s)$ (or $Q(s)$) can be used to directly influence the sensitivity functions.

For instance, using Eqs.(28)-(29):

Minimize nominal sensitivity $S(s)$:

$$Q_u(s) = \frac{L_(s)}{B_o(s)} F_1(s)$$

Minimize complementary sensitivity $T(s)$:

$$Q_u(s) = -\frac{P_(s)}{A_o(s)} F_2(s)$$

where $F_1(s)$, $F_2(s)$ are close to 1 at frequencies where $\|S(s)\|$ and $\|T(s)\|$ need to be decreased.

Internal model principle in states space

Method 1 Disturbance estimate

Suppose that disturbance enters a state space system:

$$\dot{x} = Ax + B(u + d)$$
$$y = Cx$$

Assume that disturbance $d(t)$ is unknown, but we know that it satisfies some differential equations. This implies that $d(t)$ is generated by an exo-system.

$$\dot{x}_d = A_d x_d$$
$$d = C_d x_d$$

Since,

$$D(s) = C_d(sI - A_d)^{-1}x_d(0) = C_d\frac{Adj(sI - A_d)}{det(sI - A_d)}x_d(0)$$

where $x_d(0)$ is iniital value of $x_d(t = 0)$. Thus, the disturbance generating polynomial is nothing but the characteristic polynomial of A_d,

$$\Gamma_d(s) = det(sI - A_d)$$

For example, if $d(t)$ is a sinusoidal signal,

$$\begin{pmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_{d1} \\ x_{d2} \end{pmatrix}$$
$$d = x_{d1}$$

The characteristic polynomial, as expected, is:

$$\Gamma_d(s) = det(sI - A_d) = s^2 + \omega^2$$

If we *knew* $d(t)$ then an obvious control is:

$$u = -d + v - Kx$$

where K is the state feedback gain. However, $d(t)$ is generally unknown. Thus, we estimate it using an observer. First, augment the plant model.

$$\begin{pmatrix} \dot{x} \\ \dot{x}_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix}$$

Notice that the augmented system is not controllable from u. Nevertheless, if d has effect on y, it is observable from y.

Thus, we can design an observer for the augmented system, and use the observer state for feedback:

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ x_d\end{pmatrix} = \begin{pmatrix}A & BC_d\\ 0 & A_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix} + \begin{pmatrix}B\\ 0\end{pmatrix}u - \begin{pmatrix}L_1\\ L_2\end{pmatrix}(y - C\hat{x})$$

$$u = -C_d\hat{x}_d + v - K\hat{x} = v - \begin{pmatrix}K & C_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix}$$

where $L = [L_1^T, L_2^T]^T$ is the observer gain. The controller can be simplified to be:

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ x_d\end{pmatrix} = \begin{pmatrix}A - BK - L_1C & 0\\ -L_2C & A_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix} - \begin{pmatrix}-B & L_1\\ 0 & L_2\end{pmatrix}\begin{pmatrix}v\\ y\end{pmatrix}$$

$$u = -\begin{pmatrix}K & C_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix} + v$$

The $y(t) \to u(t)$ controller transfer function $G_{yu}(s)$ has, as poles, eigenvalues of $A - BK - L_1C$ and of A_d. Since $\Gamma_d(s) = det(sI - A_d)$ the disturbance generating polynomial, the controller has $\Gamma_d(s)$ in its denominator.

This is exactly the **Internal Model Principle**.

Method 2: Augmenting plant dynamics

In this case, the goal is to introduce the disturbance generating polynomial into the controller dynamics by filtering the output $y(t)$. Let $\dot{x}_d = A_d x_d$, $d = C_d x_d$ be the disturbance model.

Nominal plant and output filter:

$$\dot{x} = Ax + Bu + Bd$$
$$y = Cx$$
$$\dot{x}_a = A_d x_a + C_d^T y(t)$$

Stabilize the augmented system using (observer) state feedback:

$$u = -[K_o \quad K_a] \begin{pmatrix} \hat{x} \\ x_a \end{pmatrix}$$

where $\hat{x}$ is the observer estimate of the original plant itself.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

Notice that x_a need not be estimated since it is generated by the controller itself!

The transfer function of the controller is: $C(s) =$

$$\begin{pmatrix} K_o & K_a \end{pmatrix} \begin{pmatrix} sI - A + BK + LC & BK \\ 0 & sI - A_d \end{pmatrix}^{-1} \begin{pmatrix} L \\ C_d^T \end{pmatrix}$$

from which it is clear that the its denominator has $\Gamma_d(s) = det(sI - A_d)$ in it. i.e. the Internal Model Principle.

An **intuitive** way of understanding this approach:

For concreteness, assume that the disturbance $d(t)$ is a sinusoid with frequency ω.

- Suppose that the closed loop system is stable. This means that for any bounded input, any internal signals will also be bounded.

- For the sake of contradiction, if some residual sinusoidal response in $y(t)$ still remains:

$$Y(s) = \frac{\alpha(s, 0)}{s^2 + \omega^2}$$

- The augmented state is the filtered version of $Y(s)$,

$$U(s) = -K_a X_a(s) = \frac{K_a \alpha(s, 0)}{(s^2 + \omega)^2}$$

The time response of $x_a(t)$ is of the form

$$x_a(t) = \gamma sin(\omega t + \phi_1) + \delta \cdot t \cdot sin(\omega t + \phi_2)$$

The second term will be unbounded.

- Since $d(t)$ is a bounded sinusoidal signal, $x_a(t)$ must also be bounded. This must mean that $y(t)$ does not contain sinusoidal components with frequency ω.

The most usual case is to combat constant disturbances using integral control. In this case, the augmented state is:

$$x_a(t) = \int_0^t y(\tau) d\tau.$$

It is clear that if the output converges to some steady value, $y(t) \to y_\infty$, y_∞ must be 0. Or otherwise $x_a(t)$ will be unbounded.