Your Name: ____________________ Your Signature: ____________________

- **Exam duration:** 2 hours and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place [a box around your final answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- You can ask as many questions as you want.
- This exam has 18 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	15	
2	15	
3	30	
4	10	
5	25	
6	10	
7	20	
8	15	
9	20	
10	20	
Total	180	

1. (15 total points) Assume that $\dot{x} = Ax$ is an asymptotically stable continuous-time LTI system.

   For each of the following statements, determine if it is true or false. If it is true, explain why; if it is false, find a counter example.

   (a) (3 points) The system $\dot{x}(t) = -Ax(t)$ is asymptotically stable.
   Solutions: False. Eigenvalues of $-A$ are $-\lambda_i, \forall i = 1, \dots, n$, which are all in the RHP.

   (b) (3 points) The system $\dot{x}(t) = A^\top x(t)$ is asymptotically stable.
   Solutions: True. Eigenvalues of $A^\top$ are the same as A.

   (c) (3 points) The system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable (assume A^{-1} exists).
   Solutions: True. Eigenvalues of A^{-1} are the same as $\frac{1}{\lambda_i}$ and they're all in the open LHP.

   (d) (3 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.
   Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}$.

   (e) (3 points) The system $\dot{x}(t) = A^2 x(t)$ is asymptotically stable.
   Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. (15 total points) Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system.

   For each of the following statements, determine if it is true or false. If it is true, explain why; if it is false, find a counter example.

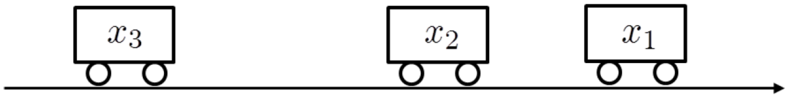
   (a) (3 points) The system $x(k+1) = -Ax(k)$ is asymptotically stable.
   Solutions: True. Eigenvalues remain in the unit disk.

   (b) (3 points) The system $x(k+1) = A^\top x(k)$ is asymptotically stable.
   Solutions: True. Eigenvalues do not change.

   (c) (3 points) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume A^{-1} exists).
   Solutions: False. Eigenvalues becomes larger than 1.

   (d) (3 points) The system $x(k+1) = (A + A^\top)x(k)$ is asymptotically stable.
   Solutions: False. Counter example: $A = 0.9$.

   (e) (3 points) The system $x(k+1) = A^2x(k)$ is asymptotically stable.
   Solutions: True. If $-1 < \lambda_i < 1$, $Rightarrow\ 0 < \lambda_i^2 < 1$.



3. (30 total points) Consider three cars moving on the same lane, whose initial locations at time $t = 0$ are $x_1(0) = x_2(0) = x_3(0) = 0$. The above figure exemplifies the movement of cars in 1-D. Suppose the goal is for all three cars to meet at the same location (it does not matter where this meet-up location is). To achieve this goal, the following system dynamics can be designed, where $u(t)$ is an input control for the leading car:

$$\dot{x}_1(t) = x_2(t) - x_1(t) + u(t) \quad (1)$$

$$\dot{x}_2(t) = \frac{x_1(t) + x_3(t)}{2} - x_2(t) \quad (2)$$

$$\dot{x}_3(t) = x_2(t) - x_3(t) \quad (3)$$

In other words, the leading and trailing cars will both move toward the middle car instantaneously; while the middle car will move towards the center of the leading and the trailing cars. The derivative of each individual state is obviously the velocity.

(a) (5 points) Represent the above dynamics of the three controls as an LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A, B are matrices that you should determine.

Solutions: Clearly, $A = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) (5 points) Find e^{At} for all $t \in \mathbb{R}$.

Solutions: $e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

(c) (15 points) Suppose $u(t) = 1, t \geq 0$. Find the expression of $x(t)$ for $t \geq 0$.

Solution: Given that $u(t) = 1 \; \forall t \geq 0$, we obtain:

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & & \\ & e^{-2(t-\tau)} & \\ & & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5(1-e^{-2t}) & \\ & & 1-e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \end{aligned}$$

(d) (3 points) Describe the steady-state behaviors of $x_i(t), i = 1,2,3$. Your description must have physical, applied meaning.

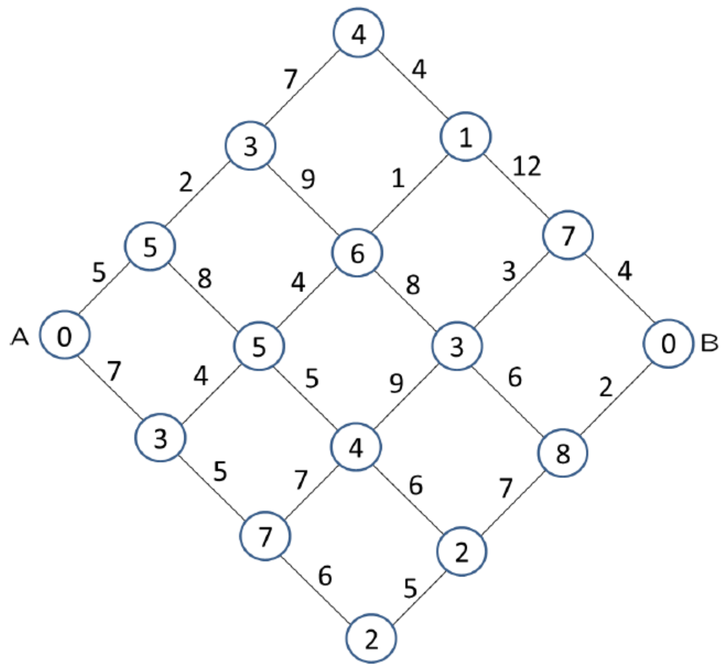
Solutions: If $t \to \infty$, we obtain:

$$x(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.25t + 0.625 \\ 0.25t - 0.125 \\ 0.25t - 0.375 \end{bmatrix}$$

In the steady state, all three cars move at the constant velocity 0.25 to the right, with car 1 still in the lead and car 3 still trailing, and the distance between car 1 and car 2 is 0.75 while the distance between car 2 and car 3 is 0.25.

(e) (2 points) What is the constant velocity for the three cars?

Solutions: 0.25.

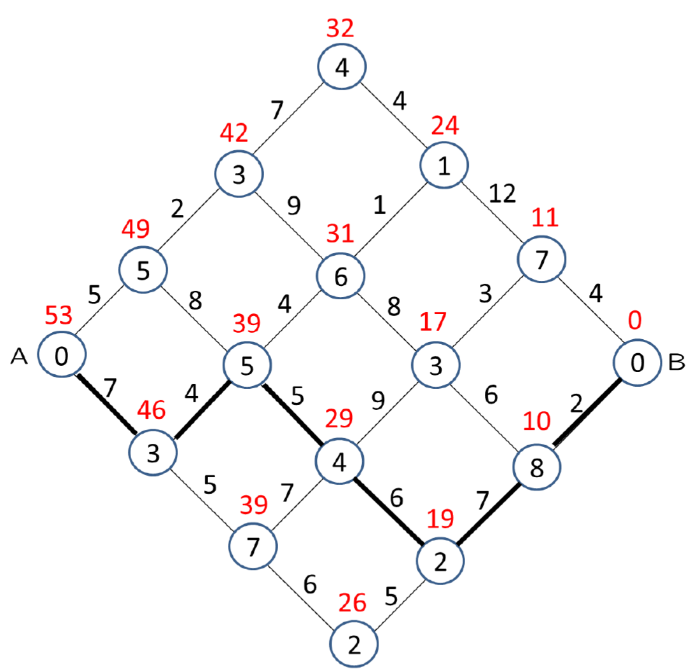


4. (10 total points) Suppose in the above graph, one starts from node A on the left and tries to reach node B on the right by only moving to the right at each step. The cost of any path is the sum of the following:

- the cost of all the edges it passes through as indicated by the numbers above the edges
- the cost of all the intermediate nodes it visits as indicated by the numbers inside the circles.

(a) (10 points) Use the dynamic programming method to find the path from A to B with the smallest cost. You should only use dynamic programming to solve the above problem. You receive **no credit** if you don't show how you applied DP.

Solutions:



5. (25 total points) Consider the following CT-LTI model of a dynamical system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \;\; x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

with the following cost function:

$$J = \int_0^\infty \left(x^\top x + u^2 \right) dt.$$

(a) (20 points) Find the linear state-feedback control law that minimizes J.

(b) (5 points) Find the value of the performance index for the closed-loop system.

6. (10 total points) Compute $x = x(t)$ that solves the following optimal control problem:

$$\text{minimize } J(u) = \int_1^2 \sqrt{1 + u^2(t)}dt$$

$$\text{subject to} \quad \dot{x} = u, \ x(1) = 0, \ x(2) = 5.$$

(a) (10 points) Find the linear state-feedback control law that minimizes J.

7. (20 total points) For the following dynamical system under unknown inputs,

$$\dot{x}_p = A_p x_p + B_p^{(1)} u_1 + B_p^{(2)} u_2$$
$$y = C_p x_p,$$

a sliding-mode observer (SMO) can be designed with the following dynamics:

$$\begin{aligned} \dot{\hat{x}}_p &= A_p \hat{x}_p + B_p^{(1)} u_1 + L(y - \hat{y}) - B_p^{(2)} E(\hat{y}, y, \eta) \\ \hat{y} &= C_p \hat{x}_p, \end{aligned}$$

where $E(\cdot)$ is defined as (η is SMO gain):

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \dfrac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2}, & \text{if } F(\hat{y} - y) \neq 0 \\ 0, & \text{if } F(\hat{y} - y) = 0. \end{cases}$$

(a) (10 points) The SMO design objective is to find matrices $P = P^\top, F$ and L that satisfy the following equations:

$$FC_p = (B_p^{(2)})^\top P$$
$$(A_p - LC_p)^\top P + P(A_p - LC_p) = -Q$$

Are the above two equations linear matrix inequalities? If no, formulate the above equations as a set of linear matrix inequalities.

(b) (10 points) Write a CVX script to solve the SMO design problem, written as an LMI.

8. (15 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\dot{x}_p = A_p x_p + B_p \hat{u} \tag{4}$$
$$y = C_p x_p \tag{5}$$
$$\dot{x}_c = A_c x_c + B_c \hat{y} \tag{6}$$
$$u = C_c x_c + D_c \hat{y}, \tag{7}$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

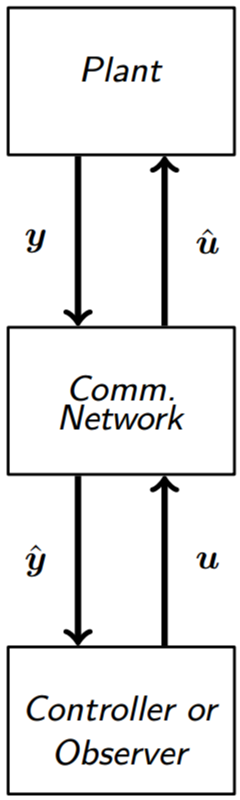


Figure 1: NCS architecture.

The network-induced error state is defined as follows:

$$e = \begin{bmatrix} e_y \\ e_u \end{bmatrix} = \begin{bmatrix} \hat{y} - y \\ \hat{u} - u \end{bmatrix}.$$

(a) (15 points) Derive the dynamics of the combined NCS state:

$$z = \begin{bmatrix} x_p \\ x_c \\ e_y \\ e_u \end{bmatrix},$$

i.e., derive

$$\boxed{\dot{z} = Az}$$

where A is a matrix you should determine in terms of $A_p, B_p, C_p, A_c, B_c, C_c, D_c$ only, given that ZOH is considered for signals $\hat{y}$ and $\hat{u}$.

9. (20) The following optimization problem is given:

$$\underset{x}{\text{minimize}} \frac{x^\top Q x}{x^\top P x},$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hint: set the denominator to 1, and make this a constraint to the optimization problem. Then, you'll get an equality-constrained optimization problem, and hence you can use the KKT conditions we discussed in class.

(a) (20 points) Solve the above optimization problem.

10. (20) The following optimization problem is given:

$$\text{maximize} \quad x_1^2 + 4x_2^2$$

$$\text{subject to} \quad x_1^2 + 2x_2^2 \leq 2$$

(a) (10 points) Derive the KKT conditions and find the set of points satisfying these conditions.

(b) (10 points) Apply the second order necessary conditions to determine whether the KKT conditions-satisfying points are minimizers or not.