Your Name:

Your Signature:

- **Exam duration:** 3 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- Do not stress, take a deep breath, and solve whatever problems you can.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the back of the pages and indicate that you have done so.
- You can ask as many questions as you want.
- This exam has 29 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules. Good luck!

Question Number	Maximum Points	Your Score
1	15	
2	50	
3	50	
4	30	
5	25	
6	35	
7	45	
8	25	
9	25	
Total	300	

1. (15 total points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. For each of the following statements, determine if it is true or false. If it is true, **prove** why; if it is false, find a counter example.

   (a) (3 points) The system $\dot{x}(t) = -2Ax(t)$ is asymptotically stable.

   (b) (3 points) The system $\dot{x}(t) = (A^\top)^{-1}x(t)$ is asymptotically stable.

   (c) (3 points) The system $\dot{x}(t) = -A^{-1}x(t)$ is asymptotically stable.

   (d) (3 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

   (e) (3 points) The system $\dot{x}(t) = A^2x(t)$ is asymptotically stable.

2. (50 total points) Answer the following miscellaneous questions.

   (a) (10 points) In networked control systems, there are two types of controls. Control *of* networks, and control *over* the networks. Explain the differences between the two. Give examples.

(b) (10 points) A function $\phi(x)$ is Globally Lipschitz with Lipschitz constant L if

$$\|\boldsymbol{\phi}(x_1) - \boldsymbol{\phi}(x_2)\| \leq L\|x_1 - x_2\|, \;\; L \geq 0.$$

Find the Lipschitz constant for the following function $\phi(x) = x^4$, if $x \in [-2,2]$. You will have to use the triangular inequality and the following hint.
Hint: $b^4 - a^4 = (b-a)(b^3 + b^2a + ba^2 + a^3)$

(c) (10 points) Represent the following inequality as a linear matrix inequality, given that A, b, θ are given quantities:

$$||Ax - b|| \leq \theta.$$

Hint: you should square both sides, and then use Schur complements.

(d) (10 points) Using Schur complements, represent the following inequalities as a single big LMI, where A, B, Q, R are given matrices and P is the LMI variable:

$$A^\top PA + Q - P - A^\top PB(R + B^\top PB)^{-1}B^\top PA \succ 0, \quad P = P^\top \succ 0.$$

Hint: Schur complements are cute.

(e) (10 points) Let $\mathcal{S}$ be the set of positive semi-definite matrices. Prove that $\mathcal{S}$ is a convex set.

3. (50 total points) The objective of this problem is to show you how LMIs are nothing but nonlinear (but still convex) optimization problems. You are given the following optimization problem:

$$\begin{aligned} \textbf{OP1:} \quad & \text{minimize} \quad && \textbf{trace}(P) = p_1 + p_2 \\ & \text{subject to} \quad && AP + PA^\top + Q = 0 \qquad (1) \\ & && P = P^\top \succ 0 \qquad (2) \end{aligned}$$

where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. For the above problem, assume that $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the optimization variable. In other words, you have three variables to solve for, since P is symmetric and positive definite.

(a) (10 points) Define a new variable $x = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top$ and write the first constraint in **OP1** as a linear system of equations, i.e., $\tilde{A}x = b$, where $\tilde{A} \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^{4\times 1}$ are matrices you should determine.

(b) (5 points) Write the second positive definiteness constraint on P (i.e., $P = P^\top \succ 0$) as a nonlinear set of equations. Remember that a matrix is positive definite if and only if all of its leading principal minors are positive. You should obtain two inequality constraints here.

(c) (10 points) Using the above transformations, write **OP1** as an simple optimization problem with a linear cost function, linear equality constraints, and quadratic inequality constraints. You should get something like this:

$$\begin{aligned} \textbf{OP2} \equiv \textbf{OP1} \quad \underset{x}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & \tilde{A}x = b && (3) \\ & x_1 > 0 && (4) \\ & x^\top Q x + x^\top \tilde{b} + c > 0 && (5) \end{aligned}$$

where $c, \tilde{A}, b, Q, \tilde{b}$, and c are constant matrices and vectors that you should have already determined.

(d) (25 points) Derive the KKT conditions for the developed optimization problem (**OP2**) and solve for p_1, p_2 and p_3 (or vector x). To know whether you solution is write, it should satisfy the constraints obviously.

4. (30 total points) Solve the following problems, given the augmented MPC dynamics,

$$\begin{aligned} x_a(k+1) &= \Phi_a x_a(k) + \Gamma_a \Delta u(k) + \Psi_a \Delta w(k) \\ y(k) &= C_a x_a(k), \end{aligned}$$

where

$$x_a \in \mathbb{R}^{n+p}, \Gamma_a \in \mathbb{R}^{n+p \times m}, C_a \in \mathbb{R}^{p \times n+p}.$$

and $\Delta w(k)$ is the rate of change of disturbances that are all assumed to be known for all k.

(a) (10 points) For a predicted horizon N_p, derive an equation that relates the predicted outputs to $x_a(k)$ and the MPC variables Δu. That is, derive matrices W, Z, M in this equation:

$$Y = \begin{bmatrix} y(k+1|k) \\ y(k+2|k) \\ \cdots \\ y(k+N_p|k) \end{bmatrix} = W x_a(k) + Z \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix} + M \begin{bmatrix} \Delta w(k) \\ \Delta w(k+1) \\ \vdots \\ \Delta w(k+N_p-1) \end{bmatrix}$$

$$= W x_a(k) + Z \Delta U + M \Delta W.$$

(b) (10 points) Derive the optimal ΔU^* if the given cost function is (without constraints):

$$J(\Delta U) = \frac{1}{2}(r - Y)^\top Q(r - Y) + \frac{1}{2}\Delta U^\top R \Delta U, \; Q = Q^\top \succ 0, R = R^T \succ 0.$$

(c) (10 points) Suppose that you're given the following constraints on the rate of change of the control action:

$$u^{\min} \leq \Delta U \leq u^{\max}.$$

Write the corresponding optimization problem in the following form:

$$\begin{array}{ll} \text{minimize} & J(\Delta U) \\ \text{subject to} & g(\Delta U) \leq 0\,, \end{array}$$

where $g(\Delta U)$ is a linear set of constraints.

5. (25 total points) Consider the following dynamic system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

We are interested in designing an **output** feedback controller of the form

$$u(t) = Fy(t)$$

where F is the matrix variable.

(a) (25 points) Your design should insure that the closed loop system is asymptotically stable. Design matrix F using linear matrix inequalities and then **write down the corresponding CVX code.**

You are almost halfway through this unnecessarily long exam...

...It's a good time to remember that although you're trying so hard to ace this exam, we're all doomed...

...Climate change, rising sea levels, e-scooters, the Justin Biebers of the world...

However, and in short, if you do well on this exam, you are in a way decreasing the Worldsuck. Actually, there is a foundation to decrease Worldsuck. Consider donating.

6. (35 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\dot{x}_p = A_p x_p + B_p \hat{u} \tag{6}$$
$$y = C_p x_p \tag{7}$$
$$\dot{x}_c = A_c x_c + B_c \hat{y} \tag{8}$$
$$u = C_c x_c + D_c \hat{y}, \tag{9}$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

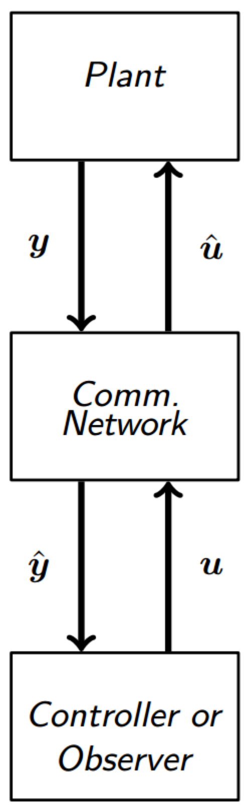

Figure 1: NCS architecture.

Assume that the network effect is modeled as pure time-delay to the exchanged signals, i.e.,

$$\hat{y}(t) = y(t - \tau), \quad \hat{u}(t) = u(t - \tau).$$

Define $x(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}$ to be the augmented state of the networked control system.

(a) (15 points) Obtain the closed-loop dynamics of time-delay based NCS:

$$\dot{x}(t) = \Psi_0 x(t) + \Psi_1 x(t - \tau).$$

In other words, you'll have to derive Ψ_0 and Ψ_1.

(b) (15 points) Using the **second order** Taylor series expansion of $x(t-\tau)$:

$$x(t-\tau) \approx \sum_{n=0}^{2} (-1)^n \frac{\tau^n}{n!} x^{(n)}(t)$$

and the fact that $\dot{x}_p(t-\tau) = \dot{x}_c(t-\tau) = 0$ for NCSs, obtain the closed loop dynamics of this form:

$$\dot{x}(t) = \Gamma(\tau, \tau^2)x(t),$$

where $\Gamma(\tau, \tau^2)$ is a matrix which is a function of τ and τ^2 that you should determine.

(c) (5 points) What happens when $\tau = 0$? In high-level terms, analyze the stability of the NCS as τ increases.

7. (45 total points) Given the following plant dynamics:

$$\dot{x}_p = A_p x_p + B_p^{(1)} u_1 + B_p^{(2)} u_2$$
$$y = C_p x_p, \ x_p(0) \text{ not given}$$

where $u_2(t)$ is the unknown input vector. The system consists of n states, m_1 known inputs, m_2 unknown inputs, and p measurable outputs. We want to design a dynamic unknown input observer (UIO) which takes the following form:

$$\dot{x}_c = A_c x_c + B_c^{(1)} y + B_c^{(2)} u_1,$$
$$\hat{x}_p = x_c + My,$$

The UIO is motivated by writing x_p as:

$$x_p = (I - MC_p)x_p + MC_p x_p = x_c + My.$$

Assume that the updated x_c takes the following form:

$$x_c = (I - MC_p)x_p.$$

(a) (15 points) Find $\dot{x}_c = A_c x_c + B_c^{(1)} y + B_c^{(2)} u_1$, where $A_c, B_c^{(1)}, B_c^{(2)}$ are matrices that you should determine, assuming that the unknown input vector is nullified and a convergence term is added to x_c, as discussed in class.

(b) (15 points) Write an ODE script to simulate the plant and and the designed observer.

(c) (15 points) If your state-estimates are converging, how can you reconstruct (or estimate) the vector of unknown inputs from the system dynamics? You only need this equation. Think about it—the solution is very simple.

8. (25 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p \hat{u} + B_w w && (10) \\ y &= C_p x_p && (11) \\ \dot{x}_c &= A_c x_c + B_c \hat{y} + B_v v && (12) \\ u &= C_c x_c + D_c \hat{y}, && (13) \end{aligned}$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

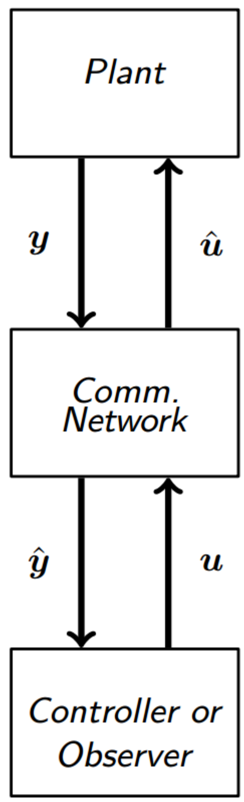

Figure 2: NCS architecture.

The network-induced error state is defined as follows:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix}.$$

(a) (25 points) Derive the dynamics of the combined NCS state:

$$z(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \\ e_y(t) \\ e_u(t) \end{bmatrix},$$

i.e., derive

$$\boxed{\dot{z}(t) = Az(t) + Ba(t)}$$

where A, B are matrices you should determine in terms of $A_p, B_p, B_w, C_p, A_c, B_c, B_v, C_c, D_c$ only, given that ZOH is considered for signals $\hat{y}$ and $\hat{u}$, and $a(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$.

9. (25 points) The following optimization problem is given:

$$\underset{x=[x_1\, x_2]^\top}{\text{minimize}} \frac{x^\top Q x}{x^\top P x},$$

where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Hint: set the denominator to 1, and make this a constraint to the optimization problem. Then, you'll get an equality-constrained optimization problem, and hence you can use the KKT conditions we discussed in class.

(a) (25 points) Solve the above minimization problem.