EE 5243 **Final Examination — Ahmad F. Taha** **Spring 2017**

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Your Signature:

- **Exam duration:** 3 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place $\boxed{\text{a box around your final answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- You can ask as many questions as you want.
- This exam has 28 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	15	—
2	40	−10
3	55	—
4	30	—
5	30	−12
6	35	—
7	45	−15
8	25	—
9	25	−2
Total	300	261/300

1. (15 total points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. For each of the following statements, determine if it is true or false. If it is true, **prove** why; if it is false, find a counter example.

(a) (3 points) The system $\dot{x}(t) = -Ax(t)$ is asymptotically stable.

The eigenvalues of A are $Av = \lambda v \iff 0 = (A - \lambda I)v \iff A - \lambda I = 0 \iff A = \lambda I$

since A is stable, then $\lambda_i < 0, \forall i$.

Now, if we substitute A with $-A$, then $-Av = \lambda v \iff 0 = (-A - \lambda I)v \iff -A - \lambda I = 0 \iff -A = \lambda I$ or $A = -\lambda I$. In this form, we have $-\lambda_i > 0$ (because $\lambda_i < 0$); hence $\boxed{\dot{x}(t) = -Ax(t) \text{ is not stable.}}$

(b) (3 points) The system $\dot{x}(t) = A^\top x(t)$ is asymptotically stable.

By using eigenvalues decomposition, we have

$A = T\Lambda T^{-1} \iff A^T = T\Lambda^T T^{-1}$, but since $\Lambda = \Lambda^T$, which is a diagonal matrix consists of all eigenvalues of A, and since A is stable. $\boxed{\text{the } A^T \text{ is also stable.}}$

(c) (3 points) The system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable.

We have

$A = T\Lambda T^{-1} \iff A^{-1} = T\Lambda^{-1}T^{-1}$, where $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & & \vdots \\ 0 & \cdots & & \frac{1}{\lambda_n} \end{bmatrix}$

since $\lambda_i < 0$, then $\frac{1}{\lambda_i} < 0 \ \forall i$, thus $\boxed{\dot{x}(t) = A^{-1}x(t) \text{ is stable.}}$

(d) (3 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, and $A + A^T = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

Since $\det(A) = 1$, but $\det(A + A^T) = 4 - 6 = -2$, which is not stable

$\boxed{\text{Thus } \dot{x}(t) = (A + A^T)x(t) \text{ is not stable}}$ with $\boxed{A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$

(e) (3 points) The system $\dot{x}(t) = A^2x(t)$ is asymptotically stable.

We have

$A = T\Lambda T^{-1} \iff A^2 = T\Lambda^2 T^{-1}$

$\boxed{\begin{array}{l} \text{Since } \lambda_i < 0 \text{, then } \lambda_i^2 > 0 \text{ which is not stable} \\ \text{Example } A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ then } A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}}$

2. (40 total points) Answer the following miscellaneous questions.

   (a) (10 points) In networked control systems, there are two types of controls. Control *of* networks, and control *over* the networks. Explain the differences between the two. Give examples.

Control of networks: in this paradigm, we design a controller along with the communication networks. Hence, the controller not only controls the plant, but also control the networks.

Example: ?

Control over the networks: in this paradigm, we design a controller which is able to overcome the effect, such as disturbances and time delays, which is caused by the network.

Example: ?

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(b) (10 points) Define the separation principle we discussed in class. Give a simple, high-level example.

The separation principle states that, the design of a controller and an observer can be done independently.

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(c) (10 points) Represent the following inequality as a linear matrix inequality, given that A, b, θ are given quantities:

$$||Ax - b|| \leq \theta.$$

Hint: you should square both sides, and then use Schur complements.

$$||Ax - b|| \leq \theta$$
$$||Ax - b||^2 \leq \theta^2$$
$$(Ax-b)^T(Ax-b) \leq \theta^2$$
$$(x^TA^T - b^T)(Ax-b) \leq \theta^2$$
$$0 \leq \theta^2 - (x^TA^T - b^T)(Ax-b)$$

by using Schur complement, we obtain

$$\boxed{\begin{bmatrix} \theta^2 & x^TA^T - b^T \\ Ax-b & 1 \end{bmatrix} \succeq 0}$$

(d) (10 points) Using Schur complements, represent the following inequalities as a single big LMI, where A, B, Q, R are given matrices and P is the LMI variable:

$$A^\top PA + Q - P - A^\top PB(R + B^\top PB)^{-1}B^\top PA \succ 0, \quad P = P^\top \succ 0$$

By assuming $R + B^TPB \succ 0$, then applying Schur complement yields

$$\begin{bmatrix} A^TPA + Q - P & A^TPB \\ B^TPA & R + B^TPB \end{bmatrix} \succ 0, \quad P = P^T \succ 0$$

Combining both, we obtain

$$\boxed{\begin{bmatrix} A^TPA + Q - P & A^TPB & 0 \\ B^TPA & R + B^TPB & 0 \\ 0 & 0 & P \end{bmatrix} \succ 0}$$

3. (55 total points) The objective of this problem is to show you how LMIs are nothing but nonlinear (but still convex) optimization problems. You are given the following optimization problem:

$$\begin{aligned} \textbf{OP1: } \text{minimize} \quad & \textbf{trace}(P) = p_1 + p_2 \\ \text{subject to} \quad & AP + PA^\top + Q = 0 \quad (1) \\ & P = P^\top \succ 0 \quad (2) \end{aligned}$$

where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. For the above problem, assume that $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the optimization variable. In other words, you have three variables to solve for, since P is symmetric and positive definite.

(a) (15 points) Define a new variable $x = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top$ and write the first constraint as a linear system of equations, i.e., $\tilde{A}x = b$, where $\tilde{A} \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^{4\times 1}$ are matrices you should determine.

$\tilde{A} \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^{3\times 1}$

$$AP + PA^T + Q = 0$$

$$\begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -P_1 - 2P_2 & -P_2 - 2P_3 \\ -2P_2 & -2P_3 \end{bmatrix} + \begin{bmatrix} -P_1 - 2P_2 & -2P_2 \\ -P_2 - 2P_3 & -2P_3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2P_1 - 4P_2 & -3P_2 - 2P_3 \\ -3P_2 - 2P_3 & -4P_3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

we obtain

$$\begin{aligned} -2P_1 - 4P_2 + 2 &= 0 \\ -3P_2 - 2P_3 &= 0 \\ -4P_3 + 2 &= 0 \end{aligned} \iff \begin{aligned} -2P_1 - 4P_2 &= -2 \\ -3P_2 - 2P_3 &= 0 \\ 4P_3 &= 2 \end{aligned} \iff \begin{bmatrix} -2 & -4 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

if $x = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$, then

$$\boxed{\tilde{A}x = b \iff \begin{bmatrix} -2 & -4 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}}$$

(b) (5 points) Write the second positive definiteness constraint on P as a nonlinear set of equations. Remember that a matrix is positive definite if and only if all of its leading principal minors are positive. You should obtain two inequality constraints here.

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \succ 0 \text{ , here } \boxed{P_1 > 0 \text{ and } P_1P_3 - P_2^2 > 0}$$

or, if $x = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$, we have

$$\boxed{x_1 > 0 \text{ and } x_1x_3 - x_2^2 > 0}$$

(c) (10 points) Using the above transformations, write **OP1** as an simple optimization problem with a linear cost function, linear equality constraints, and quadratic inequality constraints. You should get something like this:

$$\begin{aligned} \textbf{OP2} \equiv \textbf{OP1} \quad \underset{x}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & \tilde{A}x = b & (3) \\ & x_1 > 0 & (4) \\ & x^\top Qx + x^\top \tilde{b} + c > 0 & (5) \end{aligned}$$

where $c, \tilde{A}, b, Q, \tilde{b}$, and c are constant matrices and vectors that you should have already determined.

$$\underset{x}{\text{Minimize}} \quad \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Subject to} \quad \begin{bmatrix} -2 & -4 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$x_1 > 0$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 > 0$$

(d) (25 points) Derive the KKT conditions for the developed optimization problem (**OP2**) and solve for p_1, p_2 and p_3.

$$\begin{aligned} &\min \ c^T x \\ &\text{subject to } \ \tilde{A}x = b \\ &\qquad x_1 > 0 \\ &\qquad x^T Q x + x^T \tilde{b} + c > 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\min \ c^T x \\ &\text{subject to } \ \tilde{A}x - b = 0 \\ &\qquad -x^T Q x - x^T \tilde{b} - c < 0 \\ &\qquad d^T x < 0 \ , \text{where } d^T = [-1 \ 0 \ 0] \end{aligned}$$

Lagrangian

$$L(x, \mu, \lambda) = c^T x + \mu^T \begin{bmatrix} -x^T Q x - x^T \tilde{b} - c \\ d^T x \end{bmatrix} + \lambda^T (\tilde{A} x - b)$$

$$= c^T x + \mu_1 (-x^T Q x - x^T \tilde{b} - c) + \mu_2 d^T x + \lambda^T (\tilde{A} x - b)$$

$$\nabla_x L(x, \mu, \lambda) = c + \mu_1(-2Qx - \tilde{b}) + \mu_2 d + \lambda^T \tilde{A} = -2\mu_1 Q x - \mu_1 \tilde{b} + \mu_2 d + \lambda^T \tilde{A} + c$$

such that

$$\nabla_x L(x^*, \mu^*, \lambda^*) = -2\mu_1 Q x - \mu_1 \tilde{b} + \mu_2 d + \lambda^T \tilde{A} + c = 0$$

$$= [\mu_1 \ \mu_2] \begin{bmatrix} -2Qx - \tilde{b} \\ d \end{bmatrix} + \lambda^T \tilde{A} + c = 0$$

if $\mu > 0$, then $\begin{bmatrix} -2Qx - \tilde{b} \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which contradicts the d, hence it is not possible

if $\mu = 0$, then $d < 0$ and $-2Qx - \tilde{b} < 0$

also $\lambda^T \tilde{A} + c = 0 \Leftrightarrow \lambda^T \tilde{A} = -c$

$$[\lambda_1 \ \lambda_2 \ \lambda_3] \begin{bmatrix} -2 & -4 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2\lambda_1 - 4\lambda_2 & & \\ & -3\lambda_2 - 2\lambda_3 & \\ & & 4\lambda_3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

we obtain $\lambda_3 = 0$

$-3\lambda_2 - 0 = -1 \Leftrightarrow \lambda_2 = 1/3$

$-2\lambda_1 - 4/3 = -1 \Leftrightarrow 2\lambda_1 = -1/3 \Leftrightarrow \lambda_1 = -1/6$

if $\mu_1 > 0$
$\mu_2 = 0$, then $-2Qx - \tilde{b} = 0$

$$-2\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2x_3 \\ 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we have $x_3 = 0$ and $x_2 = 0$ (not a solution)

let $\tilde{A}x - b = 0 \iff$ we obtain $x_3 = 1/2$

$x_2 = -1/3$

$2x_1 = 2 - 4(-1/3)$

$x_1 = 1 + 2/3 = 5/3$

hence we get $P = \begin{bmatrix} 5/3 & -1/3 \\ -1/3 & 1/2 \end{bmatrix}$

because $-2\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2(1/2) \\ 2(-1/3) \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2/3 \\ 0 \end{bmatrix} \leq 0$

It satisfies the KKT.

$\boxed{\text{Thus } p_1 = 5/3,\ p_2 = -1/3,\ p_3 = 1/2}$

↳ This solution is ~~partially~~ correct.

4. (30 total points) Solve the following problems, given the augmented MPC dynamics,

$$
\begin{aligned}
x_a(k+1) &= \Phi_a x_a(k) + \Gamma_a \Delta u(k) + \Psi_a \Delta w(k) \\
y(k) &= C_a x_a(k),
\end{aligned}
$$

where

$$x_a \in \mathbb{R}^{n+p}, \Gamma_a \in \mathbb{R}^{n+p \times m}, C_a \in \mathbb{R}^{p \times n+p}.$$

and $\Delta w(k)$ is the rate of change of disturbances that are all assumed to be known for all k.

(a) (10 points) For a predicted horizon N_p, derive an equation that relates the predicted outputs to $x_a(k)$ and the MPC variables Δu. That is, derive matrices W, Z, M in this equation:

$$
Y = \begin{bmatrix} y(k+1|k) \\ y(k+2|k) \\ \cdots \\ y(k+N_p|k) \end{bmatrix} = W x_a(k) + Z \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix} + M \begin{bmatrix} \Delta w(k) \\ \Delta w(k+1) \\ \vdots \\ \Delta w(k+N_p-1) \end{bmatrix}
$$

$$= W x_a(k) + Z\Delta U + M \Delta W.$$

We observe that

$$x_a(k+1|k) = \Phi_a x_a(k) + \Gamma_a \Delta u(k) + \Psi_a \Delta w(k)$$

$$
\begin{aligned}
x_a(k+2|k) &= \Phi_a x_a(k+1|k) + \Gamma_a \Delta u(k+1) + \Psi_a \Delta w(k+1) \\
&= \Phi_a(\Phi_a x_a(k) + \Gamma_a \Delta u(k) + \Psi_a \Delta w(k)) + \Gamma_a \Delta u(k+1) + \Psi_a \Delta w(k+1) \\
&= \Phi_a^2 x_a(k) + \Phi_a \Gamma_a \Delta u(k) + \Gamma_a \Delta u(k+1) + \Phi_a \Psi_a \Delta w(k) + \Psi_a \Delta w(k+1)
\end{aligned}
$$

hence

$$
\begin{bmatrix} x_a(k+1|k) \\ x_a(k+2|k) \\ \vdots \\ x_a(k+N_p|k) \end{bmatrix} = \begin{bmatrix} \Phi_a \\ \Phi_a^2 \\ \vdots \\ \Phi_a^{N_p} \end{bmatrix} x_a(k) + \begin{bmatrix} \Gamma_a & 0 & \cdots & 0 \\ \Phi_a \Gamma_a & \Gamma_a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_a^{N_p-1}\Gamma_a & \Phi_a^{N_p-2}\Gamma_a & \cdots & \Gamma_a \end{bmatrix} \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix}
$$

$$
+ \begin{bmatrix} \Psi_a & 0 & \cdots & 0 \\ \Phi_a \Psi_a & \Psi_a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_a^{N_p-1}\Psi_a & \Phi^{N_p-2}\Psi_a & \cdots & \Psi_a \end{bmatrix} \begin{bmatrix} \Delta w(\ ) \\ \Delta w(k+1) \\ \vdots \\ \Delta w(k+N_p-1) \end{bmatrix}
$$

since $y(k+i|k) = C_a x_a(k+i|k)$, then we have

(b) (10 points) Derive the optimal ΔU^* if the given cost function is (without constraints):

$$J(\Delta U) = \frac{1}{2}(r - Y)^\top Q(r - Y) + \frac{1}{2}\Delta U^\top R\Delta U,\ Q = Q^\top \succ 0, R = R^T \succ 0.$$

Since $Y = Wx_a(k) + Z\Delta U + M\Delta W$, then

$$J(\Delta U) = \frac{1}{2}(r - Wx_a(k) - Z\Delta U - M\Delta W)^T Q (r - Wx_a(k) - Z\Delta U - M\Delta W) + \frac{1}{2}\Delta U^T R \Delta U$$

the optimal ΔU^* is obtained when $\frac{\partial J(\Delta U)}{\partial \Delta U} = 0$, thus

$$\frac{\partial J(\Delta U)}{\partial \Delta U} = \frac{1}{2}(-Z^T)Q(r - Wx_a(k) - Z\Delta U - M\Delta W) + \frac{1}{2}(r - Wx_a(k) - Z\Delta U - M\Delta W)^T Q(-Z) + R\Delta U$$

$$0 = -Z^T Q(r - Wx_a(k) - Z\Delta U - M\Delta W) + R\Delta U$$

$$0 = -Z^T Q(r - Wx_a(k) - M\Delta W) + Z^T Q Z \Delta U + R\Delta U$$

$Z^T Q(r - Wx_a(k) - M\Delta W) = (Z^T Q Z + R)\Delta U$, since $Q \succ 0$ and $R \succ 0$, then $(Z^T Q Z + R)^{-1}$ does exist. Thus

$$\boxed{(Z^T Q Z + R)^{-1} Z^T Q (r - Wx_a(k) - M\Delta W) = \Delta U^*}$$

(c) (10 points) Suppose that you're given the following constraints on the rate of change of the control action:

$$u^{\min} \leq \Delta U \leq u^{\max}.$$

Write the corresponding optimization problem in the following form:

$$\begin{array}{ll} \text{minimize} & J(\Delta U) \\ \text{subject to} & g(\Delta U) \leq 0\,, \end{array}$$

where $g(\Delta U)$ is a linear set of constraints.

$$U^{min} \leq \Delta U \leq U^{max} \iff \begin{bmatrix} -\Delta U \\ \Delta U \end{bmatrix} \preceq \begin{bmatrix} -U^{min} \\ U^{max} \end{bmatrix} \iff \begin{bmatrix} -I \\ I \end{bmatrix} \Delta U \preceq \begin{bmatrix} -U^{min} \\ U^{max} \end{bmatrix}$$

observe that we can have $g(\Delta U) = \begin{bmatrix} -I \\ I \end{bmatrix} \Delta U - \begin{bmatrix} -U^{min} \\ U^{max} \end{bmatrix} \preceq 0$, hence

the optimization problem becomes

$$\boxed{\begin{array}{ll} \text{minimize} & \frac{1}{2}(r-Y)^T Q (r-Y) + \frac{1}{2}\Delta U^T R \Delta U \\ \text{subject to} & \begin{bmatrix} -I \\ I \end{bmatrix} \Delta U - \begin{bmatrix} -U^{min} \\ U^{max} \end{bmatrix} \preceq 0 \end{array}}$$

5. (30 total points) Consider the following CT-LTI model of a dynamical system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t), \ \ x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

with the following cost function:

$$J = \pi \int_0^\infty \left(2x^\top(t)x(t) + \sqrt{2}u^2(t) \right) dt.$$

(a) (25 points) Find the linear state-feedback control law that minimizes J.

Observe that $J = \pi \int_0^\infty 2x^T(t)x(t) + \sqrt{2}u^2(t)\, dt$

$$= \int_0^\infty [x_1\ x_2] \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t)(\pi\sqrt{2})u(t)\, dt$$

$$= \int_0^\infty x^T x + u^T R u\, dt \quad \text{with } Q = \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix} \text{ and } R = \pi\sqrt{2}$$

The Riccati equation is, for $P = P^T = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} > 0$

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$$Q + A^TP + PA - PBR^{-1}B^TP = 0$$

$$\begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\left(\frac{1}{\pi\sqrt{2}}\right)[0\ -1]\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_1 - P_2 & P_2 - P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_1 - P_2 \\ P_2 & P_2 - P_3 \end{bmatrix} - \begin{bmatrix} -P_2 \\ -P_3 \end{bmatrix}\left(\frac{1}{\pi\sqrt{2}}\right)[-P_2\ -P_3] = 0$$

$$\begin{bmatrix} 2\pi + 2P_1 & P_1 \\ P_1 & 2\pi + P_2 - P_3 \end{bmatrix} - \frac{1}{\pi\sqrt{2}}\begin{bmatrix} P_2^2 & P_2P_3 \\ P_2P_3 & P_3^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2\pi + 2P_1 - \frac{1}{\pi\sqrt{2}}P_2^2 & P_1 - \frac{1}{\pi\sqrt{2}}P_2P_3 \\ P_1 - \frac{1}{\pi\sqrt{2}}P_2P_3 & 2\pi + P_2 - P_3 - \frac{1}{\pi\sqrt{2}}P_3^2 \end{bmatrix} = 0$$

$P_1 = \frac{1}{\pi\sqrt{2}}P_2P_3$

$2\pi + P_2 - P_3 - \frac{1}{\pi\sqrt{2}}P_3^2 = 0$

$2\pi + 2P_1 - \frac{1}{\pi\sqrt{2}}P_2^2 = 0$

if $P_2 = 1$, then $P_1 = \frac{1}{\pi\sqrt{2}}P_3 \iff P_3 = \pi\sqrt{2}P_1$

and $2\pi + 2P_1 - \frac{1}{\pi\sqrt{2}} = 0$ and $P_3 = \frac{1}{2} - \pi^2\sqrt{2}$

$2P_1 = \frac{1}{\pi\sqrt{2}} - 2\pi$

$P_1 = \frac{1}{2\pi\sqrt{2}} - \pi$

hence

$$\boxed{P = \begin{bmatrix} \frac{1}{2\pi\sqrt{2}} - \pi & 1 \\ 1 & \frac{1}{2} - \pi^2\sqrt{2} \end{bmatrix}}$$

P should be positive definite

$u(t) = -Kx(t) = -R^{-1}B^T P x(t)$

$= \dots$

(b) (5 points) Find the value of the performance index for the closed-loop system.

$$J = x_0^T P x_0$$

$$J = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\pi\sqrt{2}} - \pi & 1 \\ 1 & \frac{1}{2} - \pi^2\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{1}{\pi\sqrt{2}} - 2\pi + 1 & \frac{5}{2} - \pi^2\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$J = \frac{2}{\pi\sqrt{2}} - 4\pi + 2 + \frac{5}{2} - \pi^2\sqrt{2}$$

$$\boxed{J = \frac{2}{\pi\sqrt{2}} - 4\pi + \frac{9}{2} - \pi^2\sqrt{2}}$$

-2

6. (35 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p \hat{u} && (6) \\ y &= C_p x_p && (7) \\ \dot{x}_c &= A_c x_c + B_c \hat{y} && (8) \\ u &= C_c x_c + D_c \hat{y}, && (9) \end{aligned}$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

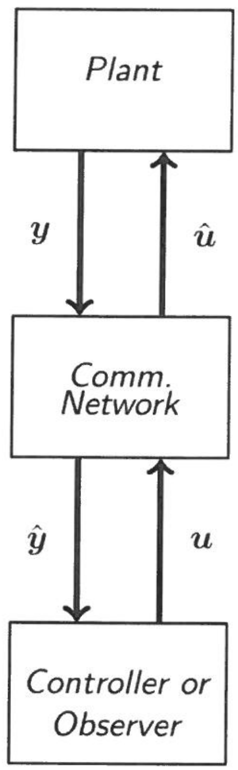

Figure 1: NCS architecture.

Assume that the network effect is modeled as pure time-delay to the exchanged signals, i.e.,

$$\hat{y}(t) = y(t-\tau), \quad \hat{u}(t) = u(t-\tau).$$

Define $x(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}$ to be the augmented state of the networked control system.

(a) (15 points) Obtain the closed-loop dynamics of time-delay based NCS:

$$\dot{x}(t) = \Psi_0 x(t) + \Psi_1 x(t-\tau).$$

In other words, you'll have to derive Ψ_0 and Ψ_1.

Plant dynamics:

$$\dot{x}_p = A_p x_p + B_p \hat{u}$$
$$y = C_p x_p$$

Controller dynamics:

$$\dot{x}_c = A_c x_c + B_c \hat{y}$$
$$u = C_c x_c + D_c \hat{y}$$

since

$$u(t) = C_c x_c(t) + D_c \hat{y}(t) = C_c x_c(t) + D_c y(t-\tau) = C_c x_c(t) + D_c C_p x_p(t-\tau)$$
$$\hat{y}(t) = y(t-\tau) = C_p x_p(t-\tau)$$

we now have

$$\dot{x}_p(t) = A_p x_p(t) + B_p(C_c x_c(t) + D_c C_p x_p(t-\tau)) = A_p x_p(t) + B_p C_c x_c(t) + B_p D_c C_p x_p(t-\tau)$$
$$\dot{x}_c(t) = A_c x_c(t) + B_c C_p x_p(t-\tau)$$

if $x(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}$, then

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A_p & B_p C_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_p D_c C_p & 0 \\ B_c C_p & 0 \end{bmatrix} \begin{bmatrix} x_p(t-\tau) \\ x_c(t-\tau) \end{bmatrix}$$

such that $\dot{x}(t) = \Psi_0 x(t) + \Psi_1 x(t-\tau)$, where

$$\boxed{\Psi_0 = \begin{bmatrix} A_p & B_p C_c \\ 0 & A_c \end{bmatrix} \quad \text{and} \quad \Psi_1 = \begin{bmatrix} B_p D_c C_p & 0 \\ B_c C_p & 0 \end{bmatrix}}$$

(b) (15 points) Using the **second order** Taylor series expansion of $x(t-\tau)$:

$$x(t-\tau) \approx \sum_{n=0}^{2} (-1)^n \frac{\tau^n}{n!} x^{(n)}(t)$$

and the fact that $\dot{x}_p(t-\tau) = \dot{x}_c(t-\tau) = 0$ for NCSs, obtain the closed loop dynamics of this form:

$$\dot{x}(t) = \Gamma(\tau, \tau^2) x(t),$$

where $\Gamma(\tau, \tau^2)$ is a matrix which is a function of τ and τ^2 that you should determine.

$$x(t-\tau) \approx \sum_{n=0}^{2} (-1)^n \frac{\tau^n}{n!} x^{(n)}(t) = x(t) - \tau \dot{x}(t) + \frac{\tau^2}{2} \ddot{x}(t)$$

from the previous calculation, we have

$$\dot{x}(t) = \Psi_0 x(t) + \Psi_1 x(t-\tau)$$

$$\frac{d\dot{x}(t)}{dt} = \ddot{x}(t) = \Psi_0 \dot{x}(t) + \Psi_1 \dot{x}(t-\tau)$$

but since $\dot{x}(t-\tau) = \begin{bmatrix} \dot{x}_p(t-\tau) \\ \dot{x}_c(t-\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have

$$\ddot{x}(t) = \Psi_0 \dot{x}(t)$$

we obtain

$$x(t-\tau) \approx x(t) - \tau \dot{x}(t) + \frac{\tau^2}{2} \Psi_0 \dot{x}(t)$$

$$x(t-\tau) \approx x(t) + \left(-\tau I + \frac{\tau^2}{2} \Psi_0\right) \dot{x}(t)$$

substitute this, and we obtain

$$\dot{x}(t) = \Psi_0 x(t) + \Psi_1 x(t-\tau) \approx \Psi_0 x(t) + \Psi_1 \left(x(t) + \left(-\tau I + \frac{\tau^2}{2} \Psi_0\right) \dot{x}(t)\right)$$

$$\dot{x}(t) = (\Psi_0 + \Psi_1) x(t) + \Psi_1 \left(-\tau I + \frac{\tau^2}{2} \Psi_0\right) \dot{x}(t)$$

$$\left(I - \Psi_1 \left(-\tau I + \frac{\tau^2}{2} \Psi_0\right)\right) \dot{x}(t) = (\Psi_0 + \Psi_1) x(t)$$

$$\dot{x}(t) = \left(I - \Psi_1 \left(-\tau I + \frac{\tau^2}{2} \Psi_0\right)\right)^{-1} (\Psi_0 + \Psi_1) x(t)$$

where $\Psi_0 + \Psi_1 = \begin{bmatrix} A_p & B_p C_c \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} B_p D_c C_p & 0 \\ B_c C_p & 0 \end{bmatrix} = \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}$

and

$$(I-\Psi_1(-\tau+\frac{\tau^2}{2}\Psi_0))^{-1} = (I - \begin{bmatrix} B_pD_cC_p & 0 \\ B_cC_p & 0\end{bmatrix}(\begin{bmatrix} -\tau I & 0 \\ 0 & -\tau I\end{bmatrix} + \frac{\tau^2}{2}\begin{bmatrix} A_p & B_pC_c \\ 0 & A_c\end{bmatrix}))^{-1}$$

$$= (\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} - \begin{bmatrix} B_pD_cC_p & 0 \\ B_cC_p & 0\end{bmatrix}(\begin{bmatrix} -\tau I+\frac{\tau^2}{2}A_p & \frac{\tau^2}{2}B_pC_c \\ 0 & -\tau I+\frac{\tau^2}{2}A_c\end{bmatrix}))^{-1}$$

$$= (\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} - \begin{bmatrix} -\tau B_pD_cC_p + \frac{\tau^2}{2}B_pD_cC_pA_p & \frac{\tau^2}{2}B_pD_cC_pB_pC_c \\ -\tau B_cC_p + \frac{\tau^2}{2}B_cC_pA_p & \frac{\tau^2}{2}B_cC_pB_pC_c\end{bmatrix})^{-1}$$

$$= \begin{bmatrix} I + \tau B_pD_cC_p - \frac{\tau^2}{2}B_pD_cC_pA_p & -\frac{\tau^2}{2}B_pD_cC_pB_pC_c \\ \tau B_cC_p - \frac{\tau^2}{2}B_cC_pA_p & I - \frac{\tau^2}{2}B_cC_pB_pC_c\end{bmatrix}^{-1}$$

hence $\dot{x}(t) = \Gamma(\tau,\tau^2)x(t)$ with

$$\boxed{\Gamma(\tau,\tau^2) = \begin{bmatrix} I + \tau B_pD_cC_p - \frac{\tau^2}{2}B_pD_cC_pA_p & -\frac{\tau^2}{2}B_pD_cC_pB_pC_c \\ \tau B_cC_p - \frac{\tau^2}{2}B_cC_pA_p & I - \frac{\tau^2}{2}B_cC_pB_pC_c\end{bmatrix}^{-1}\begin{bmatrix} A_p+B_pD_cC_p & B_pC_c \\ B_cC_p & A_c\end{bmatrix}}$$

(c) (5 points) What happens when $\tau = 0$? In high-level terms, analyze the stability of the NCS as τ increases.

When $\tau=0$, $\Gamma(\tau,\tau^2)$ becomes

$$\Gamma(0,0) = \begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix} A_p+B_pD_cC_p & B_pC_c \\ B_cC_p & A_c\end{bmatrix},$$ since $I\cdot I = I$, then

$$\Gamma(0,0) = \begin{bmatrix} A_p+B_pD_cC_p & B_pC_c \\ B_cC_p & A_c\end{bmatrix}$$

thus, when $\tau=0$, the closed loop stability is the same with the system without time delay. But, as τ increases, $\Gamma(\tau,\tilde{\tau})$ will have eigenvalues which is going to the right of the state space (complex plane), hence, at some point when τ is big enough, the closed loop dynamics become unstable.

7. (45 total points) For the following dynamical system under unknown inputs,

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t)$$
$$y(t) = Cx(t),$$

a sliding-mode observer (SMO) can be designed with the following dynamics:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_1u_1(t) + L(y(t) - \hat{y}(t)) - B_2E(\hat{y}, y, \eta)$$
$$\hat{y}(t) = C\hat{x}(t),$$

where $E(\cdot)$ is defined as (η is SMO gain):

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \dfrac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2}, & \text{if } F(\hat{y} - y) \neq 0 \\ 0, & \text{if } F(\hat{y} - y) = 0. \end{cases}$$

(a) (15 points) The SMO design objective is to find matrices $P = P^\top \succ 0, F$ and L that satisfy the following equations for a predefined $Q = Q^\top \succ 0$:

$$FC = B_2^\top P$$

$$(A - LC)^\top P + P(A - LC) = -Q$$

Are the above two equations linear matrix inequalities? If they are not, formulate the above equations as a set of linear matrix inequalities.

It is not an LMI since $(A-LC)^TP + P(A-LC) = -Q$ is bilinear, however

$$(A-LC)^TP = (A^T - C^TL^T)P = A^TP - C^TL^TP$$

and $P(A-LC) = PA - PLC$, hence the second constraint becomes

$$A^TP + PA - C^TL^TP - PLC = -Q$$

if we let $PL = Y$ such that $Y^T = L^TP$, then we have

$$A^TP + PA - C^TY^T - YC = -Q$$

and $FC = B_2^TP$

which is linear in P, F, and Y.

(b) (15 points) Write a CVX script to solve the SMO design problem, written as an LMI.

```
cvx_begin sdp
P variable(n,n) symmetric
Y variable(n,q)
F variable(m,n)
minimize(0)
subject to
  A'P+PA - C'Y' - YC == -Q;
      FC == B'P;
    P >= I;
cvx_end
```

(c) (15 points) If your state-estimates are converging, how can you reconstruct (or estimate) the vector of unknown inputs from the system dynamics? You only need this equation. Think about it—the solution is very simple.

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t)$$

Hint: You should first discretize the above dynamical system and then use $\hat{x}(t)$ to obtain $u_2(t)$.

15

8. (25 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p \hat{u} + B_w w && (10)\\ y &= C_p x_p && (11)\\ \dot{x}_c &= A_c x_c + B_c \hat{y} + B_v v && (12)\\ u &= C_c x_c + D_c \hat{y}, && (13) \end{aligned}$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

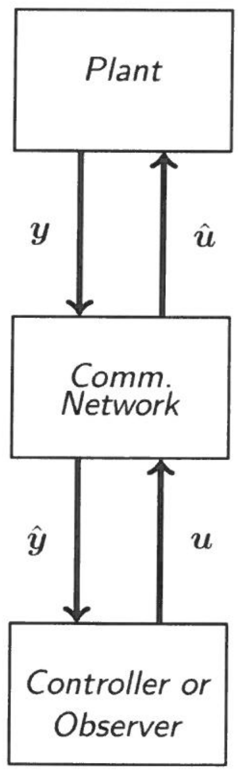

Figure 2: NCS architecture.

The network-induced error state is defined as follows:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix}.$$

(a) (25 points) Derive the dynamics of the combined NCS state:

$$z(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \\ e_y(t) \\ e_u(t) \end{bmatrix},$$

i.e., derive

$$\boxed{\dot{z}(t) = Az(t) + Ba(t)}$$

where A, B are matrices you should determine in terms of $A_p, B_p, B_w, C_p, A_c, B_c, B_v, C_c, D_c$ only, given that ZOH is considered for signals $\hat{y}$ and $\hat{u}$, and $a(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$.

Plant dynamics:

$$\dot{x}_p = A_p x_p + B_p \hat{u} + B_w w$$
$$y = C_p x_p$$

Controller dynamics:

$$\dot{x}_c = A_c x_c + B_c \hat{y} + B_v v$$
$$u = C_c x_c + D_c \hat{y}$$

since $e_u(t) = \hat{u}(t) - u(t)$, then

$$\dot{x}_p = A_p x_p + B_p (e_u + u) + B_w w$$
$$\dot{x}_p = A_p x_p + B_p e_u + B_p u + B_w w = A_p x_p + B_p e_u + B_p (C_c x_c + D_c \hat{y}) + B_w w$$

since $e_y(t) = \hat{y}(t) - y(t)$, then

$$\dot{x}_p = A_p x_p + B_p e_u + B_p C_c x_c + B_p D_c (e_y + y) + B_w w$$
$$\dot{x}_p = A_p x_p + B_p e_u + B_p C_c x_c + B_p D_c e_y + B_p D_c C_p x_p + B_w w$$
$$\dot{x}_p = (A_p + B_p D_c C_p) x_p + B_p C_c x_c + B_p D_c e_y + B_p e_u + D_w w$$

and

$$\dot{x}_c = A_c x_c + B_c (e_y + y) + B_v v = A_c x_c + B_c e_y + B_c C_p x_p + B_v v$$
$$\dot{x}_c = B_c C_p x_p + A_c x_c + B_c e_y + B_v v$$

also $\dot{e}_y(t) = \dot{\hat{y}}(t) - \dot{y}(t) = -\dot{y}(t) = -C_p \dot{x}_p(t)$

and $\dot{e}_u(t) = \dot{\hat{u}}(t) - \dot{u}(t) = -\dot{u}(t) = -C_c \dot{x}_c(t) + D_c \dot{\hat{y}}(t) = -C_c \dot{x}_c(t)$

hence $\dot{z}(t) = Az(t) + Ba(t)$ becomes

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_c(t) \\ \dot{e}_y(t) \\ \dot{e}_u(t) \end{bmatrix} = \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c & B_p D_c & B_p \\ B_c C_p & A_c & B_c & 0 \\ -C_p A_p - C_p B_p D_c C_p & -C_p B_p C_c & -C_p B_p D_c & -C_p B_p \\ -C_c B_c C_p & -C_c A_c & -C_c B_c & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_c(t) \\ e_y(t) \\ e_u(t) \end{bmatrix} + \begin{bmatrix} B_w & 0 \\ 0 & B_v \\ -C_p B_w & 0 \\ 0 & -C_c B_v \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$$

9. (25 points) The following optimization problem is given:

$$\underset{x}{\text{minimize}} \frac{x^\top Q x}{x^\top P x},$$

where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

(a) (25 points) Solve the above minimization problem.

The problem $\underset{x}{\text{minimize}} \frac{x^T Q x}{x^T P x}$ is equal to $\underset{x}{\text{minimize}}\ x^T Q x$ subject to $x^T P x - 1 = 0$

The Lagrangian is

$$L(x,\lambda) = x^T Q x + \lambda(x^T P x - 1)$$

$$\nabla_x L(x,\lambda) = 2Qx + 2\lambda P x$$

Lagrangian optimality

$$\nabla_x L(x^*,\lambda^*) = 0 \Leftrightarrow 2Qx^* + 2\lambda^* P x^* = 0 \Leftrightarrow (2Q + 2\lambda^* P)x^* = 0$$

or $Q + \lambda^* P = 0 \Leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \lambda^* \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} 2+\lambda^* & 0 \\ 0 & 3+4\lambda^* \end{bmatrix} = 0$

we have $2 + \lambda^* = 0 \Leftrightarrow \lambda_1^* = -2$ or $3 + 4\lambda^* = 0 \Leftrightarrow \lambda^* = -\frac{3}{4}$

Then, if $\lambda^* = \lambda_1^* = -2$, the we have

$$Q + \lambda_1^* P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + (-2)\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}$$

and $2(Q + \lambda_1^* P)x^* = 2\begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} x^* = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = 0 \Leftrightarrow -10x_2^* = 0 \Leftrightarrow x_2^* = 0$

and $x^T P x - 1 = [x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 = [x_1\ 4x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1$

$= x_1^2 + 4x_2^2 - 1 = 0 \Leftrightarrow$ if $x_2 = 0$, the $x_1 = \pm 1$

hence $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$

if $\lambda^* = \lambda_2^* = -\frac{3}{4}$, the we have

$$Q + \lambda_2^* P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \left(-\frac{3}{4}\right)\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & 0 \end{bmatrix}$$

and $2(Q + \lambda_2^* P)x^* = 2\begin{bmatrix} 5/4 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 5/2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff 5/2\, x_1^* = 0$

$x_1^* = 0$

and $x^T P x - 1 = x_1^2 + 4x_2^2 - 1 = 0 \iff$ if $x_1^* = 0$, then

$4x_2^{*2} - 1 = 0 \iff 4x_2^{*2} = 1 \iff x_2^{*2} = \frac{1}{4} \iff x_2^* = \pm\frac{1}{2}$

hence $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ \pm\frac{1}{2} \end{bmatrix}$

-2

Considering the cost, we have

$J = x^T Q x$. if $J_1 = x^T Q x$ for $\lambda^* = -2$, then

$J_1 = x^T Q x = (x_1 \ x_2)\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2x_1 \ 3x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 3x_2^2$

$J_1 = 2(1) + 0 = 2$

and $J_2 = 2x_1^2 + 3x_2^2$ for $\lambda^* = -\frac{3}{4}$, then

$J_2 = 2(0) + 3(\frac{1}{2})^2 = \frac{3}{2^2} = \frac{3}{4}$

Since $J_2 < J_1$, then $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \pm\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ is a candidate of local minimizer

Hessian

$$\nabla^2 L(x, \lambda^*) = 2Q + 2\lambda^* P = 2(Q + \lambda^* P)$$

if $\lambda^* = -\frac{3}{4}$ then $\nabla^2 L(x, \lambda^*) = 2\begin{bmatrix} 5/4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5/2 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$

Hence, $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \pm\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ is a local minimizer