

Module 02

CPS Background: Linear Systems Preliminaries

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In Class Assessment Exam

- **Objective:** assess students' knowledge of the course prerequisites
- You are not supposed to do so much work for this exam — it's only assessment, *remember!*
- You will receive a perfect grade whether you know the answers or not

Linear Algebra — 1

- Find the eigenvalues, eigenvectors, and inverse of matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

- **Answers**

- Eigenvalues: $\lambda_{1,2} = 5, -2$
 - Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $\mathbf{v}_2 = \begin{bmatrix} -\frac{4}{3} & 1 \end{bmatrix}^\top$
 - Inverse: $A^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$
- Write A in the matrix diagonal transformation, i.e., $A = TDT^{-1}$ — D is the diagonal matrix containing the eigenvalues of A .

- **Answers**

- Diagonal Transformation: $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1 \end{bmatrix}^{-1}$

Linear Algebra — 2

- Find the determinant, rank, and null-space set of this matrix:

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

- Answers**

- $\det(B) = 0$

- $\text{rank}(B) = 2$

- $\text{null}(B) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall \alpha \in \mathbb{R}$

- Is there a relationship between the determinant and the rank of a matrix?

- Answer**

- Yes! Matrix drops rank if determinant = zero \rightarrow minimal of 1 zero value

- True or False?

- $AB = BA$ for all A and B

- A and B are invertible $\rightarrow (A + B)$ is invertible

LTI Systems — 1

- LTI dynamical system can be represented as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_{\text{initial}} = \mathbf{x}_{t_0}, \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \quad (2)$$

- $\mathbf{x}(t)$: dynamic state-vector of the LTI system, $\mathbf{u}(t)$: control input-vector
- $\mathbf{y}(t)$: output-vector and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are constant matrices
- What is the closed-form, state solution to the above differential equation for any time-varying control input, i.e., $\mathbf{x}(t) = ?$

- **Answer:**

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_{t_0} + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

Quiz Time

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_{t_0} + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

Can you verify that the above solution is actually correct? Hint:

$$\underbrace{\frac{d}{d\theta} \left(\int_{a(\theta)}^{b(\theta)} f(x, \theta) dx \right) = \int_{a(\theta)}^{b(\theta)} \partial_{\theta} f(x, \theta) dx + f(b(\theta), \theta) \cdot b'(\theta) - f(a(\theta), \theta) \cdot a'(\theta)}_{\text{Leibniz Differentiation Theorem}}$$

LTI Systems — 2

- What are the poles of the above dynamical system? Define asymptotic stability. What's the difference between marginal and asymptotic stability?

- **Answers:**

- Poles: $\text{eig}(A)$
- Asymptotic stability: poles **strictly** in the LHP
- Marginal stability: some poles can be on the imaginary axis

- What is the transfer function $\left(H(s) = \frac{Y(s)}{U(s)} \right)$ of the above system?

- **Answers:**

- TF:

$$H(s) = C(sI - A)^{-1}B + D$$

- Above TF valid for MIMO systems — it becomes a TF matrix, rather than a scalar quantity

- See [Chen \[1995\]](#) for more details

LTI Systems — 3

- What is the state-transition matrix for the above system?
- **Answer:**
 - $\phi(t, t_0) = e^{\mathbf{A}(t-t_0)}$
 - How about linear, **time-varying** systems?
- Under what conditions is the above dynamical system controllable? Observable? Stabilizable? Detectable?
- **Answers:**
 - Controllability
 - Observability
 - Stabilizability
 - Detectability

Optimal Control and Dynamic Observers — 1

- Define the linear quadratic regulator problem for LTI systems in both, words and equations

- Answers:**

- Objective: minimize the total cost of state-deviation and consumed control (i.e., taking control actions)
- Constraints: state-dynamics, control inputs, initial conditions
- Equations:

$$\underset{\mathbf{u}(t), \mathbf{x}(t)}{\text{minimize}} \quad J = \mathbf{x}^\top(t_1) \mathbf{F}(t_1) \mathbf{x}(t_1) + \int_{t_0}^{t_1} (\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} + 2\mathbf{x}^\top \mathbf{N} \mathbf{u}) dt$$

$$\text{subject to} \quad \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \quad (3)$$

$$\mathbf{x}(t_i) = \mathbf{x}_{t_i}$$

$$\mathbf{u} \in \mathcal{U}, \mathbf{x} \in \mathcal{X}$$

Optimal Control and Dynamic Observers — 2

- What is the optimal solution to an optimal control problem? What does it physically mean for CPSs?
- **Answers:**
 - It's the optimal trajectory of the control input and the corresponding state-trajectory
 - Physical meaning: *you're better off selecting this control input, among all other — possibly infinite — control input alternatives*
- What is a generic dynamic observer? Luenberger observer?
- **Answer:**
 - An estimator for internal states of the system

Optimization — 2

- What is an optimization problem? An unfeasible, feasible solution, an optimal solution to a generic optimization problem?
- What is a convex optimization problem? Define it rigorously

- **Answers:**

- An optimization problem of finding some $\mathbf{x}^* \in \mathcal{X}$ such that:

$$f(\mathbf{x}^*) = \min\{f(\mathbf{x}) : \mathbf{x} \in \mathcal{X}\}$$

- * $\mathcal{X} \subset \mathbb{R}^n$ is the feasible set and $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective, is called convex if \mathcal{X} is a closed convex set and $f(\mathbf{x})$ is convex on \mathbb{R}^n
- Alternatively, convex optimization problems can be written as:

$$\text{minimize} \quad f(\mathbf{x}) \tag{4}$$

$$\text{subject to} \quad g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \tag{5}$$

- * $f, g_1, \dots, g_m : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex

- See [\[Boyd & Vandenberghe, 2004\]](#) for more

Optimization — 3

- What is a semi-definite program (SDP)? Define it rigorously.
- **Answers:**
 - A semidefinite program minimizes a linear cost function of the optimization variable $z \in \mathbb{R}^n$ subject to a matrix inequality condition
 - An SDP can be formulated as follows:

$$\underset{z \in \mathbb{R}^n}{\text{minimize}} \quad f(z) \tag{6}$$

$$\text{subject to} \quad F(z) \succeq \mathbf{O}, \tag{7}$$

* Where

$$F(z) \triangleq F_0 + \sum_{i=1}^n z_i F_i$$

- Given the following optimization problem,

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && c(\mathbf{x}) \\ & \text{subject to} && \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \\ & && \mathbf{g}(\mathbf{x}) = \mathbf{0}, \end{aligned} \tag{8}$$

what are the corresponding Karush–Kuhn–Tucker (KKT) conditions?

Questions And Suggestions?



Thank You!

Please visit

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IFF you want to know more 😊

References I

Boyd, S., & Vandenberghe, L. (2004). *Convex Optimization*. New York, NY, USA: Cambridge University Press.

Chen, C.-T. (1995). *Linear System Theory and Design*. New York, NY, USA: Oxford University Press, Inc., 2nd ed.