

Module 03

Linear Systems Theory: Necessary Background

Ahmad F. Taha

EE 5243: Introduction to Cyber-Physical Systems

Email: ahmad.taha@utsa.edu

Webpage: <http://engineering.utsa.edu/~taha/index.html>



April 5, 2016

Module 03 Outline

We will review in couple lectures the necessary background needed in linear systems theory and design. Outline of this module is as follows:

- 1 Computation of solution for an ODE, LTI systems
- 2 Stability of linear systems and Jordan blocks
- 3 Discrete dynamical systems
- 4 Controllability, observability, stabilizability, detectability
- 5 Design of controllers and observers
- 6 Linearization of nonlinear systems

LTI Systems

- LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0}, \quad (1)$$

$$y(t) = Cx(t) + Du(t), \quad (2)$$

- We now know that the solution is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

- Clearly the output solution is:

$$y(t) = \underbrace{C \left(e^{A(t-t_0)}x_{t_0} \right)}_{\text{zero input response}} + \underbrace{C \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau + Du(t)}_{\text{zero state response}}$$

- **Question:** how do I analytically compute the solution to (1)?
- **Answer:** you need to (a) integrate and (b) compute matrix exponentials (given $A, B, C, D, x_{t_0}, u(t)$)

Matrix Exponential — 1

- Exponential of scalar variable:

$$e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \dots$$

- Power series converges $\forall a \in \mathbb{R}$
- How about matrices? For $A \in \mathbb{R}^{n \times n}$, matrix exponential:

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots$$

- What if we have a time-variable?

$$e^{tA} = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} = I_n + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \frac{(tA)^4}{4!} + \dots$$

Matrix Exponential Properties

For a matrix $A \in \mathbb{R}^{n \times n}$ and a constant $t \in \mathbb{R}$:

- 1 $Av = \lambda v \Rightarrow e^{At}v = e^{\lambda t}v$
- 2 ${}^1\det(e^{At}) = e^{(\text{trace}(A))t}$
- 3 $(e^{At})^{-1} = e^{-At}$
- 4 $e^{A^T t} = (e^{At})^T$
- 5 If A, B commute, then: $e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$
- 6 $e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}$

¹Trace of a matrix is the sum of its diagonal entries.

When Is It Easy to Find e^A ? Method 1

Well...Obviously if we can directly use $e^A = I_n + A + \frac{A^2}{2!} + \dots$

Three cases:

- A is nilpotent², i.e., $A^k = 0$ for some k . Example: $A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$

- A is idempotent, i.e., $A^2 = A$. Example: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

- A is of rank one: $A = uv^T$ for $u, v \in \mathbb{R}^n$

$$A^k = (v^T u)^{k-1} A, \quad k = 1, 2, \dots$$

²Any triangular matrix with 0s along the main diagonal is nilpotent

Method 2 — Jordan Canonical Form

- All matrices, whether diagonalizable or not, have a Jordan canonical form:
 $A = TJT^{-1}$, then $e^{At} = Te^{Jt}T^{-1}$

- Generally, $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$, $J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \Rightarrow$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{n_i-1} e^{\lambda_i t}}{(n_i-1)!} \\ 0 & e^{\lambda_i t} & \ddots & \frac{t^{n_i-2} e^{\lambda_i t}}{(n_i-2)!} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \Rightarrow e^{At} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_p t} \end{bmatrix} T^{-1}$$

- Jordan blocks and marginal stability

Example 1

- Find $e^{A(t-t_0)}$ for matrix A given by:

$$A = TJT^{-1} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}$$

- Solution:**

$$\begin{aligned} e^{A(t-t_0)} &= Te^{J(t-t_0)}T^{-1} \\ &= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 & 0 & 0 \\ 0 & 1 & t-t_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-(t-t_0)} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1} \end{aligned}$$

Example 2 — Quiz Time

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = 0, x(t_0) = x(1) = [0 \ 1 \ 1]^T$$

- Determine $y(t)$ if $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^+(t)$.

- Solution:** $y(t) = \begin{bmatrix} 0.5(t-1)^2 \\ t-1 \end{bmatrix}$

- MATLAB demo

From CTLTI to DTLTI Systems

- Given A, B, C, D for a continuous time, LTI system, what are the the equivalent matrices for the discrete dynamics?
- For a sampling period T , the equivalent representation is:

$$\tilde{A} = e^{AT}, \tilde{B} = \left[\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B d\tau \right], \tilde{C} = C, \tilde{D} = D$$

- Dynamics: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$, $y(k) = \tilde{C}x(k) + \tilde{D}u(k)$
- MATLAB command: `[Ad,Bd]=c2d(A,B,T)`

LTI Discrete Systems

- LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \quad x_{\text{initial}} = x_{k_0}, \quad (3)$$

$$y(k) = Cx(k) + Du(k), \quad (4)$$

- We now know that the solution is given by:

$$x(k) = A^k x_{k_0} + \sum_{i=k_0}^k A^{k-1-i} Bu(i) = A^k x_{k_0} + \sum_{i=k_0}^k A^i Bu(k-1-i)$$

- Clearly the output solution is:

$$y(k) = \underbrace{C(A^k x_{k_0})}_{\text{zero input response}} + \underbrace{C \sum_{i=k_0}^k A^{k-1-i} Bu(i) + Du(k)}_{\text{zero state response}}$$

- **Question:** how do I analytically compute the solution to (3)?
- **Answer:** you need to (a) evaluate summations and (b) compute matrix powers

Discrete LTI System Example

- Consider the following time-invariant discrete dynamics:

$$x(k+1) = T \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} T^{-1} x(k) + T \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

- Determine A^k . **Solution:** $A^k = T \begin{bmatrix} 0^k & 0 \\ 0 & 0.25^k \end{bmatrix} T^{-1}$
- Find the zero-state state-response and $x(9)$ given that $u(k) = 0.5^k 1^+(k)$
- Solution?** Work it out and show it to me next time...

Controllability — 1

A CTLTI system is defined as follows:

$$\dot{x} = Ax + Bu, x(0) = x_0$$

- Over the time interval $[0, t_f]$, control input $u(t) \forall t \in [0, t_f]$ steers the state from x_0 to x_{t_f} :

$$x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{A(t-\tau)} Bu(\tau) d\tau$$

Controllability Definition

CTLTI system is controllable at time $t_f > 0$ if for any initial state and for any target state (x_{t_f}) , a control input $u(t)$ exists that can steer the system states from $x(0)$ to $x(t_f)$ over the defined interval.

- Reachable subspace: space of all reachable states
- DTLTI Controllability

Controllability — 2

Controllability Test

For a system with n states and m control inputs, the test for controllability is that matrix

$$\mathcal{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \in \mathbb{R}^{n \times nm}$$

has full row rank (i.e., $\text{rank}(\mathcal{C}) = n$).

- The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

- 1 \mathcal{C} is full rank
- 2 PBH Test: for any $\lambda \in \mathbb{C}$, $\text{rank}[\lambda I - A \quad B] = n$
- 3 Eigenvector Test: for any vector $v \in \mathbb{C}$ of A , $v^T B \neq 0$
- 4 For any $t_f > 0$, the so-called Gramian matrix is nonsingular:

$$W(t_f) = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau$$

Observability — 1

DTLTI system (n states, m inputs, p outputs):

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \tag{5}$$

$$y(k) = Cx(k) + Du(k), \tag{6}$$

- **Application:** given that A, B, C, D , and $u(k), y(k)$ are known $\forall k = 0 : 1 : k - 1$, **can we determine** $x(0)$?

- **Solution:**

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Observability — 2

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

- Or:

$$Y(k-1) = \mathcal{O}_k x(0) + \mathcal{T}_k U(k-1) \Rightarrow$$

$$\mathcal{O}_k x(0) = Y(k-1) - \mathcal{T}_k U(k-1)$$

- Since $\mathcal{O}_k, \mathcal{T}_k, Y(k-1), U(k-1)$ are all known quantities, then we can find a unique $x(0)$ iff \mathcal{O}_k is full rank

Observability Definition

DTLTI system is **observable at time k** if the initial state $x(0)$ can be uniquely determined from any given $u(0), \dots, u(k-1), y(0), \dots, y(k-1)$.

- Unobservable subspace: null-space of \mathcal{O}_k

Observability — 3

Observability Test

For a system with n states and p outputs, the test for observability is that

matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$ has full column rank (i.e., $\text{rank}(\mathcal{O}) = n$).

- The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

- 1 \mathcal{O} is full rank, system is observable
- 2 PBH Test: for any $\lambda \in \mathbb{C}$, $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$
- 3 Eigenvector Test: for any vector $v \in \mathbb{C}$ of A , $Cv \neq 0$
- 4 The matrices $\sum_{i=0}^{n-1} (A^T)^i C^T C A^i$ for the DTLTI and $\int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau$ for the CTLTI are nonsingular

Example

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is this system controllable?
- Is this system observable?
- **Answers:** Yes, Yes!
- MATLAB commands: `ctrb`, `obsv`

Controller Design

- Open-loop control: design $u(t)$ directly, i.e., through optimization or learning
- Closed-loop control: design $u(t)$ as a function of state, i.e., $u(t) = g(x, t)$
- Linear state-feedback (LSF) control: design matrix K such that the control $u(t) = -Kx(t) + v(t)$ yields a desirable state-response
- Dynamics under LSF:

$$\dot{x}(t) = (A - BK)x(t) + Bv(t), \quad v(t) \text{ is a reference signal}$$

- **Objective:** design K such that eigenvalues of $A - BK$ are stable or at a certain location
- **Fact:** if the system is **controllable**, $\text{eig}(A - BK)$ can be arbitrarily reassigned

Stabilizability

Stabilizability Definition

DTLTI or CTLIT system, defined by (A, B) , is stabilizable if there exists a matrix K such that $A - BK$ is stable.

Stabilizability Theorem

DTLTI or CTLIT system, defined by (A, B) is stabilizable if all its uncontrollable modes correspond to stable eigenvalues of A .

Facts:

- A is stable $\Rightarrow (A, B)$ is stabilizable
- (A, B) is controllable $\Rightarrow (A, B)$ is stabilizable as well
- (A, B) is not controllable \Rightarrow it could still be stabilizable

Observer Design

Original system with unknown $x(0)$:

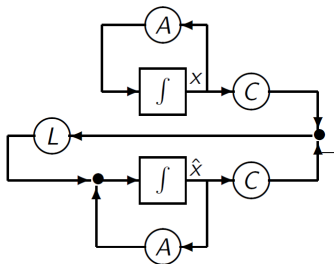
$$\dot{x} = Ax,$$

$$y = Cx$$

Simulator with linear feedback:

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \quad \hat{x}(0) = 0$$

$$\hat{y} = C\hat{x}$$



- Define dynamic estimation error: $e(t) = x(t) - \hat{x}(t)$
- Error dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t)$$

- Hence, $e(t) \rightarrow 0$, as $t \rightarrow \infty$ if $\text{eig}(A - LC) < 0$
- **Objective:** design observer/estimator gain L such that $\text{eig}(A - LC) < 0$ or at a certain location

Detectability

Detectability Definition

DTLTI or CTLIT system, defined by (A, C) , is detectable if there exists a matrix L such that $A - LC$ is stable.

Detectability Theorem

DTLTI or CTLIT system, defined by (A, C) is detectable if all its unobservable modes correspond to stable eigenvalues of A .

Facts:

- A is stable $\Rightarrow (A, C)$ is detectable
- (A, C) is observable $\Rightarrow (A, C)$ is detectable as well
- (A, B) is not observable \Rightarrow it could still be detectable
- If system has some unobservable modes that are unstable, then no gain L can make $A - LC$ stable
- \Rightarrow Observer will fail to track system state

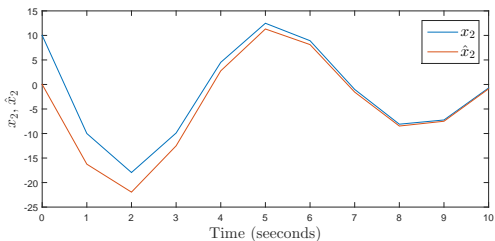
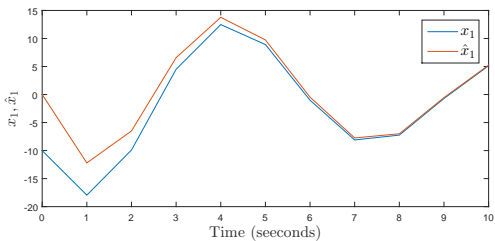
Example — Controller Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Is the system stable? What are the eigenvalues?
- **Solution:** unstable, $\text{eig}(A) = 4, -2$
- Find linear state-feedback gain K (i.e., $u = -Kx$), such that the poles of the closed-loop controlled system are -3 and -5
- Characteristic polynomial: $\lambda^2 + (k_1 - 2)\lambda + (3k_2 - k_1 - 8) = 0$
- **Solution:** $u = -Kx = -[10 \ 11] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -10x_1 - 11x_2$
- MATLAB command: $K = \text{place}(A, B, \text{eig_desired})$
- What if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, can we stabilize the system?

Example — Observer Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $C = [0.5 \quad 1]$
- Find linear state-observer gain $L = [l_1 \quad l_2]^T$ such that the poles of the estimation error are -5 and -3
- Characteristic polynomial:
$$\lambda^2 + (-2 + l_2 + 0.5l_1)\lambda + (-8 + 0.5l_2 + 2.5l_1) = 0$$
- **Solution:** $L = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
- MATLAB command: $L = \text{place}(A', C', \text{eig_desired})$

MATLAB Example

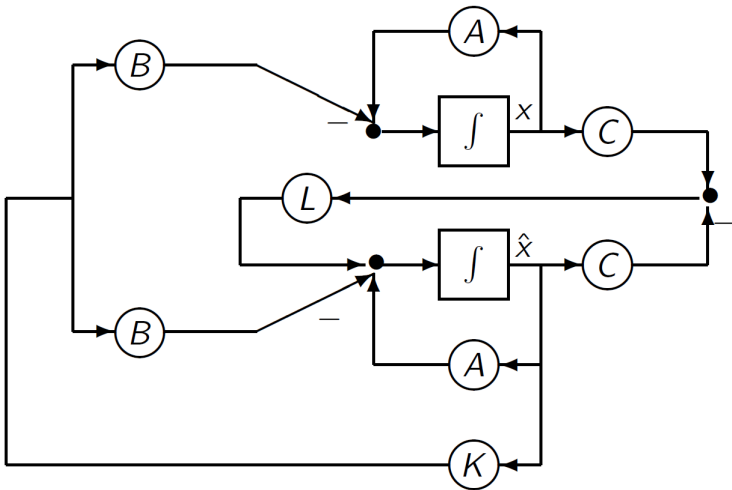


```
A=[1 -0.8; 1 0];
B=[0.5; 0];
C=[1 -1];
% Selecting desired poles
eig_desired=[.5 .7];
L=place(A',C',eig_desired)';
% Initial state
x=[-10;10];
% Initial estimate
xhat=[0;0];
% Dynamic Simulation
XX=x;
XXhat=xhat;
T=10;
% Constant Input Signal
UU=.1*ones(1,T);
for k=0:T-1,
u=UU(k+1);
y=C*x;
yhat=C*xhat;
x=A*x+B*u;
xhat=A*xhat+B*u+L*(y-yhat);
XX=[XX,x];
XXhat=[XXhat,xhat];
end
% Plotting Results
subplot(2,1,1)
plot(0:T,[XX(1,:);XXhat(1,:)]);
subplot(2,1,2)
plot(0:T,[XX(2,:);XXhat(2,:)]);
```

Observer-Based Control — 1

- Recall that for LSF control: $u(t) = -Kx(t)$
- What if $x(t)$ is not available, i.e., it can only be estimated?
- **Solution:** get \hat{x} by designing L
- Apply LSF control using \hat{x} with a LSF matrix K to both the original system and estimator
- **Question:** how to design K and L simultaneously? Poles of the closed-loop system?

Observer-Based Control — 2



Observer-Based Control — 3

- Closed-loop dynamics:

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t) \tag{7}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - \hat{y}(t)) - BK\hat{x}(t) \tag{8}$$

- Or

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

- Transformation: $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$

- Hence, we can write:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

- If the system is controllable & observable $\Rightarrow \text{eig}(A_{textcl})$ can be arbitrarily assigned by proper K and L
- What if the system is stabilizable and detectable?

Nonlinear Systems

- Many dynamical systems are not originally linear
- To analyze the system (stability, observability, controllability), we need a linearized representation of the system
- For a nonlinear dynamical system $\dot{x}(t) = f(x)$, follow this procedure to linearize:
 - 1 Put the ODE in state-space vector form
 - 2 Find all equilibrium points by solving $f(x) = 0$
 - 3 List all the possible solutions: $x_e^1, x_e^2, x_e^3, \dots$
 - 4 Find the Jacobian matrix of the nonlinear dynamics, $Df(x)$
 - 5 Linearize using Taylor series expansion:

$$\dot{x}(t) = f(x_e^i) + Df(x) \Big|_{x=x_e^i} (x - x_e^i)$$

- 6 Determine which equilibrium points are stable. If $Df(x) \Big|_{x=x_e^i} \prec 0$, the equilibrium point is locally stable.

Linearization Example

$$\dot{x}_1 = -x_1^2 + x_2$$

$$\dot{x}_2 = 3 - x_2 - x_3$$

$$\dot{x}_3 = 2 - x_3$$

- Find all the equilibrium points for the given nonlinear system and determine their corresponding local stability

- **Solution:**

$$x_e^1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \text{ asymptotically stable}$$

$$x_e^2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \text{ unstable}$$

Questions And Suggestions?



Thank You!

Please visit

engineering.utsa.edu/~taha

IFF you want to know more 😊